Black Hole index: towards an exact macroscopic counting

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References

Based on work with Atish Dabholkar, Sameer Murthy and Ashoke Sen:

- "Supersymmetric index from black hole entropy", arXiv:1009.3226
- "Counting all dyons in $\mathcal{N} = 4$ string theory", arXiv:0803.2692
- "Perturbative tests of non-perturbative counting", arXiv:0911.0586

Motivation

$$S_{BH} = S_{micro}$$

only holds in the large charge approximation.

- On the black hole side it's enough to use a two derivative action
- On the microscopic we use a Cardy formula.
- Would subleading charge corrections agree on both sides?

Quantum entropy function

Theorem

For black holes with an AdS_2 horizon, the full quantum corrected entropy is computed using AdS_2/CFT_1 correspondence (Sen)

$$d_{hor}(\vec{q}) = \left\langle e^{-iq_a \oint_\Sigma A^a} \right\rangle_{AdS_2}^{finite}$$

- Quantum black hole entropy is interpreted as the ground state of CFT1 with given charges q_a.
- Macroscopic computation requires performing AdS_2 path integral with specific boundary conditions.

Precision holography is a very difficult problem!! Easier to compute index.

Use helicity trace index in four dimensions (1/4 BPS states in $\mathcal{N} = 4$ string th)

$$B_6 = \frac{1}{6!}\mathrm{Tr}(-1)^{2J}(2J)^6$$

It can be written in terms of contributions from horizon and hair degrees of freedom

$$\begin{aligned} B_6 &= \frac{1}{6!}\mathrm{Tr}_{hor}(-1)^{2J_{hor}+2J_{hair}}(2J_{hor}+2J_{hair})^6 \\ &= \frac{1}{6!}\mathrm{Tr}_{hor}(-1)^{2J_{hor}}\mathrm{Tr}(-1)^{2J_{hair}}(2J_{hair})^6 \\ &= \sum_{q_i} B_{horizon}(q_i)B_{hair}(q-q_i) \end{aligned}$$

The factorization happens on the assumption that all fermion zero modes have support only outside the horizon.

Theorem

For a black hole with at least 4 supercharges and near horizon geometry with AdS_2 factor, closure of supersymmetry algebra requires $SU(2)$ subalgebra! (Sen)

This $SU(2)$ can be identified with a subgroup of spatial rotation. In other words it forces the horizon to be spherically symmetric!

$$J_{hor} = 0$$

$$B_{horizon} = \mathrm{Tr}(-1)^{2J_{hor}} = \mathrm{Tr}(1) = d_{hor}$$

For the horizon, index equals degeneracy. This explains why for large charges $S_{BH} = S_{micro}$.

Some difficulties in computing this index...

- It requires identifying explicitly the hair modes; it involves doing non-linear analisys which is sometimes very difficult.
- It requires computing d_{hor} by performing AdS_2 path integral

In a certain limit, we can use the power of AdS_3 to circumvent these difficulties.

Take the example of a four dimensional black hole. Horizon geometry is $AdS_2 \times S^2 \times S^1 \times \tilde{S}^1 \times \mathcal{K}$

- Let momentum along S^1 become large keeping all other charges finite
- We take the asymptotic value of the radius of S^1 to infinity and keep all the other moduli fixed.

Theorem

Black hole solution can be regarded as a BTZ black hole living in this AdS_3 space-time (Strominger, Larsen, Balasubramanian).

For a BTZ black hole in AdS_3

- The black hole entropy should be reinterpreted as the cardy formula of the holographically dual CFT_2.
- The cardy formula is expected to hold in the full quantum theory. We shall use this definition as the quantum black hole entropy!
- It reduces the problem to computing the quantum corrected central charges!!

The CFT_2 dual to the theory in the bulk of AdS_3 does not capture all the degrees of freedom of the system.
There can be exterior degrees of freedom living at the boundary or in the region between AdS_3 and asymptotic infinity.

- Index in terms of bulk and exterior modes is

$$\begin{aligned} B_6 &= \frac{1}{6!}\mathrm{Tr}(-1)^{2h_{bulk}+2h_{exterior}}(2h_{bulk}+2h_{exterior})^6 \\ &= \frac{1}{6!}\mathrm{Tr}(-1)^{2h_{bulk}}\mathrm{Tr}(-1)^{2h_{exterior}}(2h_{exterior})^6 \\ &= \sum_i B_{bulk}(q_i)B_{exterior}(q-q_i) \end{aligned}$$

where again we assume that all fermion zero modes are exterior modes.

For large momentum n, growth of both indexes B_{bulk} and $B_{exterior}$ have Cardy like formulas, from which

$$\begin{aligned} B_{bulk}(q) &\approx e^{2\pi\sqrt{\frac{c_L^{bulk}}{6}n}} \\ B_{exterior}(q) &\approx e^{2\pi\sqrt{\frac{c_{L,eff}^{exterior}}{6}n}} \end{aligned}$$

which gives for the total index the asymptotic growth,

$$B_6 \approx e^{2\pi\sqrt{\frac{c_{L,eff}^{macro}}{6}n}},\ c_{L,eff}^{macro} = c_L^{bulk} + c_{L,eff}^{exterior}$$

For the exterior modes

$$c_L^{exterior} \neq c_{L,eff}^{exterior}$$

Exterior modes contribution:

- 1+1 dimensional SCFT for the exterior modes is invariant under $(0, 4)$ supersymmetry
- $SU(2)$ R-symmetry of this superconformal theory fails to be identified with $SU(2)_R$ spatial rotation group (Ex: center of mass motion of brane)
- A priori no relation between $c_R^{exterior}$ and $k_R^{exterior}$

The quantity $c_{L,eff}^{exterior}$ is an effective central charge and controls the asymptotic growth of $B_{exterior}$.

We use AdS_3/CFT_2 correspondence$\Rightarrow$

- Lorentz Chern-Simons term in AdS_3 gives us $c^{bulk}_{grav} = c^{bulk}_{L} - c^{bulk}_{R}$(Larsen, Kraus)
- $SU(2)$ R-symmetry is identified with the $SU(2)_R$ spatial rotation group
- R-symmetry current level k_R is given in terms of the $SU(2)_R$ Chern-Simons term in the bulk (Larsen, Kraus).
- $(0,4)$ superconformal symmetry relates $c_R = 6k_R$

$$c^{bulk}_{L} = c^{bulk}_{grav} + 6k_R$$

- Even if a priori there's no relation between $c_{L,eff}^{exterior}$ and $c_L^{exterior}$, we found from many simple examples the following relation

$$
\begin{aligned}
c_{L,eff}^{exterior} &= c_{grav}^{exterior} + 6k_R^{exterior} \\
&= (c_L^{exterior} - c_R^{exterior}) + 6k_R^{exterior}
\end{aligned}
$$

- Putting both bulk and exterior pieces together we find

$$
\begin{aligned}
c_{L,eff}^{macro} &= c_L^{bulk} + c_{L,eff}^{exterior} \\
&= c_{grav}^{bulk} + c_{grav}^{exterior} + 6k_R^{bulk} + 6k_R^{exterior}
\end{aligned}
$$

Every term could possibly be computed from macroscopics.

It's difficult to compute the quantum corrected c_{grav}^{bulk} and k_R^{bulk}

- Charge dependent terms in c_{grav}^{bulk} and k_R^{bulk} come from reduction of Chern-Simons terms already existing in the ten dimensional 1PI action (Larsen, Kraus).
- Constant shifts can arise due to 1-loop quantum corrections after reduction to AdS_3(Ex: N^2 vs N^2-1 in AdS_5; Bilal, Chu)

If we were able to compute these loop corrections than we would know how to compute c_L^{bulk} and this way the quantum corrected black hole entropy.

For the asymptotic observer

$$c_{L,eff}^{macro} = c_{grav}^{asymp} + 6k_R^{asymp}$$

- $k_R^{asymp} = k_R^{bulk} + k_R^{exterior}$ is the total anomaly of the $SU(2)_R$ spatial rotation.
- $c_{grav}^{asymp} = c_L^{asymp} - c_R^{asymp} = (c_L^{bulk} + c_L^{exterior}) - (c_R^{bulk} + c_R^{exterior})$

In some cases both c_{grav}^{asymp} and k_R^{asymp} can be computed using anomaly inflow (Freed, Harvey, Minasian, Moore).

Anomaly inflow:

- In an anomaly free theory, as M-theory, the normal and tangent bundle anomalies of an M5-brane should be cancelled by the presence of Chern-Simons terms in the bulk theory (Freed, Harvey, Minasian, Moore).
- Anomalies on the brane can be used to predict the Chern-Simons terms.

Or use a scaling argument to predict the dependence on the charges of the coefficients of the Chern-Simons terms (Sen).

- coefficients of the Chern-Simons terms are quantized; these coefficients must be polynomial in the charges

$$c = Q.Q + Q + a$$

- l-loop contribution has the following scaling

$$c^l(q^{mag}_{NSNS}, \lambda^2 q^{ele}_{NSNS}, \lambda q_{RR}) = \lambda^{2-2l} c^l(q^{mag}_{NSNS}, q^{ele}_{NSNS}, q_{RR})$$

- For example for Type IIA theory we know there is a 1-loop Chern-Simon term of the from

$$\frac{1}{2\pi\alpha'} \int B_2 \wedge I_8(R)$$

- The constant shift a must come at 1-loop.

- The fact that at asymptotic infinity there's no loop correction to the coefficients of Chern-Simons terms, the effect of $c_{L,eff}^{exterior}$ is to cancell the 1-loop constant correction!
- The asymptotic growth of the total index that is controlled by $c_{L,eff}^{macro}$ is given in terms of the coefficients of the Chern-Simons terms.
- We found for four and five dimensional black holes that in the limit of n very large, the index is controlled by the coefficients of Chern-Simons terms computed at asymptotic infinity.

Macroscopic counting

Anti-Cardy limit

- In some cases a microscopic description of the black hole is not available due to the presence of NS5-branes.
- Nevertheless we were able to compute its macroscopic index using the coefficients of the Chern-Simons terms.
- Example: five dimensional black hole in Type IIA with NS5-F-p dual to D1-D5-p in Type IIB (take the limit Q_1 very large)

Some examples:

$\mathcal{M}$	limit	$\ln(d_{macro})$	$\ln(d_{micro})$
$K3$	IIB	$2\pi\sqrt{Q_1Q_5(n-\frac{J^2}{4Q_1Q_5})}$	$2\pi\sqrt{Q_1Q_5(n-\frac{J^2}{4Q_1Q_5})}$
$K3$	IIA	$2\pi\sqrt{Q_5(n+3)(Q_1-\frac{J^2}{Q_5(n-1)})}$	$2\pi\sqrt{Q_5(n+3)(Q_1-\frac{J^2}{Q_5(n-1)})}$

Table: five-dimensional black holes

If we computed blindly the degeneracy by applying a Cardy formula we would get (Ex: for D1-D5-p on $K3 \times S^1$)

$$\ln d = 2\pi\sqrt{n(Q_1 Q_5 + 2)}\,,\ (J = 0)$$

- Microscopic theory contains $4Q_1 Q_5 + 4$ bosons and $4Q_1 Q_5 + 4$ fermions from the "$\mathrm{Sym}^{Q_1 Q_5 + 1}$ part"
- Plus 4 bosons and 4 fermions from the "center of mass motion"

If we compute the index, the irregular part contributes as "-6" to the effective central charge that controls the growth of the index. Solves some puzzles (Lambert, de Wit).

We consider dyonic black holes in four-dimensional $\mathcal{N} = 4$ string theory. Take for example Heterotic string on T^6.

- U-duality group is

$$SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$$

- Dyonic states have electric and magnetic charges

$$\Gamma^i_\alpha = \begin{pmatrix} Q^i \\ P^i \end{pmatrix}, \; i = 1 \ldots 28, \; \alpha = 1,2$$

- The dyon spectrum is duality invariant

$$\Omega(\Gamma, \phi_\infty) = \Omega(\Gamma', \phi'_\infty)$$

- Some continuous T-duality invariants

$$Q^2,\ P^2,\ Q.P$$

- Important U-duality discrete invariant, also called torsion

  $I = \gcd(Q \wedge P) = \gcd(Q_i P_j - Q_j P_i)$, (Dabholkar, Gaiotto, Nampuri)

- Quarter-BPS states have $I \geq 1$.

Earlier work started by Dijkgraaf, Verlinde and Verlinde concerns the spectrum of primitive dyons, that is, $I = 1$. They conjectured that their degeneracy was captured by a Siegel modular form

$$\Omega(\Gamma) = \int d\rho d\sigma d\nu \frac{e^{-i\pi(Q^2\rho + P^2\sigma + 2Q.P\nu)}}{\Phi_{10}(\rho,\sigma,\nu)}$$

- Siegel modular form

$$\Phi_k\left[(A\tau + B)(C\tau + D)^{-1}\right] = \det(C\tau + D)^k \Phi_k(\tau),$$

$$\tau = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$$

- Analogy with 1/2-BPS states for heterotic string

$$d(Q^2) = \int d\tau \frac{e^{-i\pi Q^2\tau}}{\eta^{24}(\tau)}$$

- David and Sen considered D1-D5-KK system on Type IIB string theory on $K3 \times T^2$
- By studying the low energy dynamics of D1-D5 branes interacting with a KK monopole, they found a two dimensional microscopic SCFT with (0,4) supersymmetry and sigma model

$$\sigma(SCFT) = \sigma(\text{Sym}^{Q_1 Q_5 + 1}(K3)) \times \sigma(TN_1) \times \sigma(KK - P)$$

$$\mathcal{M}_{D1-D5} \times \text{Motion on TN}\times\text{KK excitations}$$

Microscopic precision counting

- Helicity trace for quarter-BPS states

$$\Omega(\Gamma,\phi_\infty) = \frac{1}{6!}\mathrm{Tr}_{(Q,P)}(-1)^{2J}(2J)^6$$

- For $I = 1$ (primitive dyons), David and Sen proved that

$$\Omega(\Gamma,\phi_\infty) = (-1)^{Q.P+1}\int_{\mathcal{C}_{\phi_\infty}} d\rho d\sigma d\nu \frac{e^{-i\pi(Q^2\rho+P^2\sigma+2Q.P\nu)}}{\Phi_{10}(\rho,\sigma,\nu)}$$

in agreement with previous conjecture.

- Their result correctly reproduces the Wald entropy of corresponding black holes for large charges including higher derivative corrections

$$S = \pi\sqrt{Q^2P^2 - (Q.P)^2} + \mathcal{O}(Q/P)$$

- It accounts for walls of marginal stability. Walls divide moduli space into chambers $(X, X', \ldots)$. Index is constant in a chamber but can jump when we cross a wall.
- The dependence of the index in the moduli is encoded in the choice of contour. Different contours $(\mathcal{C}, \mathcal{C}', \ldots)$ which cannot be deformed into each other without crossing poles of the partition function are in one-to-one correspondence with chambers $(X, X', \ldots)$.

- The Index for primitive dyons only depends on Q^2, $P^2$and $Q.P$. There are more T-duality invariants.

Theorem

Full set of T-duality invariants is Q^2, P^2, $Q.P$, I, $gcd(Q)$, $gcd(P)$ and $u(Q.P)$. (Banerjee, Sen)

By a $SL(2,\mathbb{Z})$ transformation, we can always bring dyon to the form

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} IQ_0 \\ P_0 \end{pmatrix}, \ \mathrm{gcd}(Q_0 \wedge P_0) = 1$$

and $\mathrm{gcd}(Q_0) = \mathrm{gcd}(P_0) = u(Q.P) = 1$.

Microscopic precision counting

- This subspace is invariant under the congruence subgroup $\Gamma^0(I) \in SL(2, \mathbb{Z})$.
- In this subspace

$$\Omega(\Gamma, \phi_\infty) = \Omega(Q^2, P^2, Q.P, I; \phi_\infty)$$

- Index is expected to exibit $\Gamma^0(I)$ symmetry explicitly.

For dyons with $I > 1$ the microscopic derivation of the underlying two-dimensional SCFT is a much more difficult problem.

- It involves studying D1-D5 system in the background of multi-KK monopole geometry.
- In multi KK monopole geometry there are 2-cycles that join position centers. There can be tensionless strings (Witten)!
- We have to understand the quantum effective theory of the multi-KK monopole.

Based on

- 4d-5d lift (it relates the index of five to four dimensional black holes) (Shih, Strominger, Yin)
- electric-magnetic duality
- Careful treatment of fermion zero modes; use modified elliptic genus (Maldacena, Moore, Strominger)

We propose a (0,4) SCFT with sigma model

$$\sigma(SCFT) \quad = \quad \mathrm{Sym}^{I}\left[\sigma(\mathrm{Sym}^{Q_1 Q_5+1}(K3)) \times \sigma(TN_1) \times \sigma(KK-P)\right]$$

(Dabholkar, JG, Murthy)

The modified elliptic genus gives the index

$$\Omega^l(Q,P) = \sum_{s|l} s\Omega^1\left(\frac{Q^2}{s^2}, \frac{Q.P}{s}, P^2\right)$$

Ω^1 is the fourier coefficient of $1/\Phi_{10}$.

- It correctly reproduces the Wald entropy ($s = 1$ contribution is dominant), in the large charge expansion
- It's invariant under $\Gamma^0(l)$.
- It reproduces wall crossing formula for semiprimitive dyons

$$\begin{pmatrix} lQ_0 \\ P_0 \end{pmatrix} \rightarrow \begin{pmatrix} lQ_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ P_0 \end{pmatrix}$$

- It passes perturbative test!

The two charge configuration (1 D1, n) can be mapped to IIA perturbative momentum-winding states! (Dabholkar, JG)

- They can have torsion $I \geq 1$.
- They have $Q^2 = P^2 = Q.P = 0$.
- Index is $\Omega^I(Q,P) = \sum_{s|I} s\Omega^1(0,0,0) = 0$
- There's no physical issue concerning the index in IIA frame. In both frames the answer is zero.

Conclusion

- We obtained exact results for the index on the macroscopic side in the limit of large charges
- Interesting connection between growth of index and coefficients of the Chern-Simons terms
- Exact results from microscopics for dyons with non-trivial values of torsion I.
- Macroscopic and microscopic results agree in the limit of large momentum.