M-theory and generalized geometry

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Outline

This is based on recent work with Malcolm Perry in arXiv1008.1763.
But there are a whole host of works on which it is based: Early work of Duff, Tseytlin, Nicolai and West and more recent work Hull and Zweibach, Waldram, Hitchin, Gualtieri amongst many many others.

Sorry to those I havn't mentioned but should have!

We are used to duality symmetrires in string and M-theory. There is the O(d,d) duality group of T-duality and the more complicated M-theory duality groups of M-theory reduced on T^d

Recall,

$$\begin{aligned} d &= 4 \qquad G = SL(5) \\ d &= 5 \qquad G = SO(5,5) \\ d &= 6 \qquad G = E_6 \\ d &= 7 \qquad G = E_7 \end{aligned} \tag{1}$$

These duality groups are manifest afte dimensional reduction but ut has long been conjectured that they are present in some way in the nonreduced theory ie. in 11d supergravity.

Our goal will be to reformulate eleven dimensional supergravity (at least its Bosonic sector) to make the duality group manifest without any dimensional reduction.

And we want the repackaging to be *geometric* in some sense.

To do this we need to repackage $g_{\mu\nu}$ and $C_{\mu\nu\rho}$ into a single geometric quantity.

The package will be the sort of generalized geometry developed by Hitchin and extended to M-theory and Hull and Waldram.

We will show how Generalized/Extended geometry comes from membrane world volume duality (as developed by Duff and Lu).

Most importantly we will show how to construct the dynamics of generalized metric ie. a Lagrangian for the generalized metric.

And from a canonical analysis of supergravity why a normal geometric action ie. Einstein Hilbert, cannot work.

Generalized geometry for T-duality.

The tangent space is extended from $\Lambda^1(M)$ to $\Lambda^1(M) \oplus \Lambda^{*1}(M)$
The metric on this generalized space is given by

$$M_{IJ} = \begin{pmatrix} g_{\mu\nu} - B^2_{\mu\nu} & B_\mu{}^\sigma \\ B^\rho{}_\nu & g^{\sigma\rho} \end{pmatrix} \tag{2}$$

This is extending just the tangent space but one could do that by extending the space itself. I will call this extended geometry though for the string it is normally called doubled geometry because the dimension of the space is doubled.

From the world sheet point of view we combine string fields and their T-duals into the same target space. Thus naturally the above metric acts on the forms:

$$dX^I = (dX^\mu, dy_\mu) \tag{3}$$

where y_μ is the string worldsheet field T-dual to X^μ.

Need to generalize this for membranes

Extend space to include *T-dual* membrane windings, $y_{\mu\nu}$ along with usual x^{μ} coordinates. No longer a simple doubling. Now the generalised tangent space is:

$$\Lambda^{1}(M) \oplus \Lambda^{*2}(M)\,. \tag{4}$$

What about the metric?

Lets return to the string and see how we can derive this.

Deriving the generalised $O(d,d)$ metric from the string world sheet (Duff).

$$L = \partial_\mu X^a \partial^\mu X^b g_{ab} + \epsilon^{\mu\nu} B_{ab} \partial_\mu X^a \partial_\nu X^b \tag{5}$$

Define

$$\mathcal{F}^a_\mu = \partial_\mu X^a \tag{6}$$

and

$$\tilde{\mathcal{G}}^{\mu a} = \epsilon^{\mu\nu} \mathcal{F}^a_\nu \tag{7}$$

Equations of Motion are:

$$\partial_\mu G^\mu_a = 0 \tag{8}$$

where

$$\mathcal{G}^\mu_1 = g_{ab}\mathcal{F}^b_\mu + B_{ab}\tilde{\mathcal{G}}^{\mu b} \tag{9}$$

The Bianchi Identity is:

$$\partial_\mu \tilde{\mathcal{G}}^{\mu a} = 0 \tag{10}$$

T-duality: use the following 1st order Lagrangian

$$L = h^{\mu\nu}\mathcal{F}_\mu^a\mathcal{F}_\nu^b + \epsilon^{\mu\nu}\mathcal{F}_\mu^a\mathcal{F}_\nu^b B_{ab} + \epsilon^{\mu\nu}\partial_\mu y_a \mathcal{F}_\nu^a \tag{11}$$

and introduce

$$\tilde{\mathcal{F}}_{\mu a} = \partial_\mu y_a \tag{12}$$

The Bianchi and equations motion are exchanged between the dual theories and the above variables are related as follows:

$$\begin{pmatrix} \mathcal{G}_{\mu a} \\ \tilde{\mathcal{G}}_\mu^d \end{pmatrix} = \begin{pmatrix} g_{ab} - B^2_{ab} & B_a{}^c \\ B^d{}_b & g^{cd} \end{pmatrix} \begin{pmatrix} \mathcal{F}_\mu^b \\ \tilde{\mathcal{F}}_{\mu c} \end{pmatrix} \tag{13}$$

Generalize this for the membrane.

$$L = \sqrt{-h}h^{\mu\nu}\partial_\mu X^a \partial_\nu X^b g_{ab} + \frac{1}{3}\epsilon^{\mu\nu\rho}\partial_\mu X^a \partial_\nu X^b \partial_\rho X^c C_{abc} - \sqrt{-h}. \tag{14}$$

A first order form is then given by:

$$\begin{aligned} L &= \sqrt{-h}h^{\mu\nu}\mathcal{F}^a_\mu \mathcal{F}^b_\nu g_{ab} + \frac{1}{3}\epsilon^{\mu\nu\rho}\mathcal{F}^a_\mu \mathcal{F}^b_\nu \mathcal{F}^c_\rho C_{abc} \\ &+ \frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho}\partial_\mu y_{ab}\mathcal{F}^a_\mu \mathcal{F}^b_\nu - \sqrt{-h}. \end{aligned}$$

We have now the dual field y_{ab} We can follow the string and determine the currents that are conserved either through equations of motion or Bianchi identities in the membrane

We can re-organize the information about the equations of motion and Bianchi identities by defining

$$\tilde{\mathcal{F}}_{\mu\ ab} = \partial_\mu y_{ab} \qquad (15)$$

and writing

$$\begin{pmatrix} \mathcal{G}_{\mu a} \\ \tilde{\mathcal{G}}_\mu^{mn} \end{pmatrix} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_a{}^{ef} C_{bef} & \frac{1}{\sqrt{2}} C_a{}^{kl} \\ \frac{1}{\sqrt{2}} C^{mn}{}_b & g^{mn,kl} \end{pmatrix} \begin{pmatrix} \mathcal{F}_\mu^b \\ \tilde{\mathcal{F}}_{\mu\ kl} \end{pmatrix}, \qquad (16)$$

where $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$ and has the effect of raising an antisymmetric pair of indices.

It is this matrix that we will take as the geometric way to combine the metric and three form, and call it the generalized metric, M_{IJ}. This agrees with the metric found by Hull, Pacheco and Waldram using other methods.

Now, we extend the space as before, to one with coordinates $Z^I = (x^a, y_{ab})$ and demand that M_{IJ} is the metric on the space.

Now, the next step is to write down a Lagrangian such that when we have the condition:

$$\partial_{y_{ab}} = 0 \qquad (17)$$

then we reproduce ordinary supergravity.

Ideally we would also have a more general *section condition*. This is known for the string where Hull and Zweibach have used closed string field theory to determine the most general section condition that is an O(d,d) invariant. Here we don't

know and it is an open question. There are some ideas of how to get ot it that I will return to later.

In order to be tractable we will work with the d=4 case of SL(5). We are far away from doing the full duality groups at this stage. For d>5 we also need to incude fivebrane winding modes too.

So decompose the eleven dimensional space into a trivial 4+7 split. We will concentrate on the 4 dimensional space and allow arbitrary dependence on the coordinates in those directions ie. no dimensional reduction.

So $a = 1..4$ which implies there are 10 y_{ab} coordinates and the total extended space is ten dimensional.

What is the Lagrangian describing the dynamics of the generized metric.
It is not the Einstein Hilbert. To see this compare with Kaluza Klien theory. We can imagaine the extended direction are KK directions.
This is precisely the KK anstaz, and so we then we know what such a metric will give.

$$R - nF^2 \tag{18}$$

where

$$F^{ab}_{cd} = \partial_c C^{ab}_d - \partial_d C^{ab}_c \tag{19}$$

This looks promising but the field strength, F is wrong, it is not gauge invariant under

$$C \rightarrow C + d\lambda \,. \tag{20}$$

This is crucial. The gauge invariances of the orginal theory mean we require the combination of diffeomorphism invariance and two form gauge transfomations to leave the Lagrangian invariant.

Interestingly, the algebra of these transformtions form a Courant algebra and it is this rather than the algebra of diffeomorphism that plays the key role.

What we mean by a scalar is something invariant under these transformations.

Out of a hat.........

We can construct the Lagrangian with all the right properties:

$$
\begin{aligned}
L &= \Bigg(\frac{1}{12} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \\
&+ \frac{1}{24} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS}) \\
&- \frac{1}{2} M^{MN} M^{PQ} (M^{RS} \partial_P M_{RS})(\partial_M M_{NQ}) \Bigg)
\end{aligned}
$$

When M_{IJ} is independent of y_{ab} then (up to surface terms)

$$L = R - \frac{1}{48} H^2 \tag{22}$$

where $H = dC$, reproducing the ordinary Lagrangian.

This is somewhat miraculous!

Note, only true up to surface terms.

What is the SL(5) invariant section condition?

What are its local symmetries?

Expect that the above two questions are related
For the string case there has been a similar construction by Hull and Zwiebach using closed string field theory. They also construct a Lagrangian for the O(d,d) metric that reproduces the usual one when there is no y_a dependence but thet also have an $O(d,d)$ invariant constraint, which is that fields must obey:

$$\partial_I \partial^I = \partial_{x^a} \partial_{y_a} = 0 \tag{23}$$

They can reproduce the Courant algebra of the Canonical supergravity.

but they must use the constraint to do so.

Since we don't have the equivalent of string field theory, this gives use our best hope of actually finding the right constraint condition in our case.

open question.

Other duality groups. eg E^6

$$\Lambda^1(M) \to \Lambda^{*2}(M) \oplus \Lambda^{*5}(M) \tag{24}$$

So we have coordinates

$$Z^I = (x^a, y_{ab}, y_{abcde}) \tag{25}$$

with $a = 1..6,\ ab = 7..21,\ abcde = 22..27$. Thus the space is 27 dimensional and the 27 is the fundamental of E_6 The y_{abcde} correspond to fivebrane winding modes.

Do the same for the fivebrane world volume theory to produce the E_6 generalized metric also produced by Pacheco and Waldram and Hull. There is one nice thing in this scheme, the fivebrane world volume theory also contains the wrapped membranes as fluxes of the world volume field strength.

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2}C_a{}^{ef}C_{bef} + C_a{}^{stuvw}C_{bstuvw} & \frac{1}{\sqrt{2}}C_a{}^{kl} & C_a{}^{stuvw} \\ \frac{1}{\sqrt{2}}C^{mn}{}_b & g^{mn,kl} & 0 \\ C^{mnopq}{}_b & 0 & g^{mnopq,stuvw} \end{pmatrix}$$

Can this method be extended to D-branes to get a generalized geometry that incorporates the RR fields, certainy seems possible.

Main problem, So far, we have not been able to produce L, need a bigger hat or a better idea.

Note, the presence of C_6 probabley means we can't really do the split we have in mind. The duality realtion between H_4 and H_7 is 11 dimensional and so we can't get away with ignoring the other space.

So far only rewritten SUGRA (Bosonic part).

Need the section condition

Fermions?

Boundary terms?

Mathemtics of generalized geoemtry ie. Connections, Curvatures, invariances etc.

Uses:

Higher derivative terms??

Blackhole duality invariant entropy formulae

Nongeometric solutions eg U-fold type goemetries

More perhaps, but need to go futher...