

# Thermodynamic Bubble Ansatz

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arXiv:0911.4708, L.F.A, D. Gaiotto & J. Maldacena

## Gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

- At large values of  $\lambda$  we can use the *AdS/CFT* duality.
- Scattering amplitudes  $\rightarrow$  minimal surfaces in *AdS*

### What we will do in practice

Compute the area of minimal surfaces in *AdS*

- 1 Mathematical problem
- 2 Minimal surfaces
  - Minimal surfaces in  $AdS_3$
  - Minimal surfaces in  $AdS_5$
- 3 Solving the problem
- 4 Conclusions and outlook

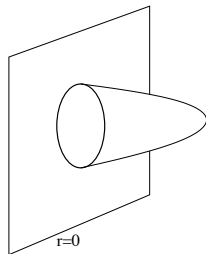
## AdS/CFT duality

Four dimensional maximally  
 SUSY Yang-Mills

$\Leftrightarrow$

Type IIB string theory on  
 $AdS_5 \times S^5$

- We can compute quantities of  $\mathcal{N} = 4$  SYM at strong coupling by doing geometrical computations on AdS, e.g. expectation value of Wilson loops at strong coupling

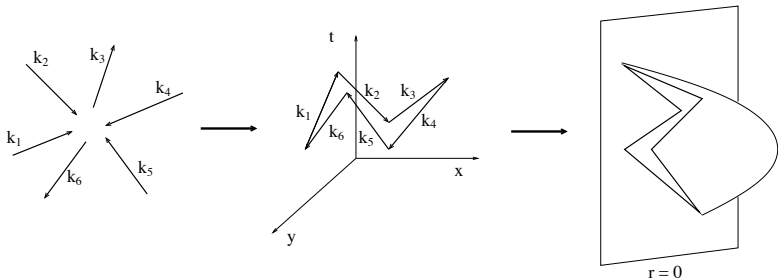


- $ds^2 = R^2 \frac{dx_{3+1}^2 + dr^2}{r^2}$
- Minimal area ending (at  $r = 0$ ) on the Wilson loop.

$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

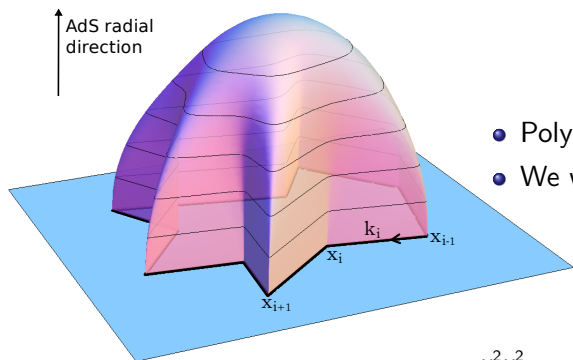
- Amplitudes at strong coupling  $\rightarrow$  Minimal area in  $AdS_5$ !

$$ds^2 = \frac{dx_3^2 - dt^2 + dr^2}{r^2}$$



$$A_N \approx e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}, \quad \lambda \gg 1$$

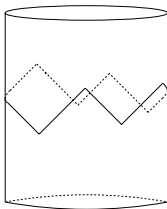
## Math problem: Minimal surfaces/Soap bubbles in AdS



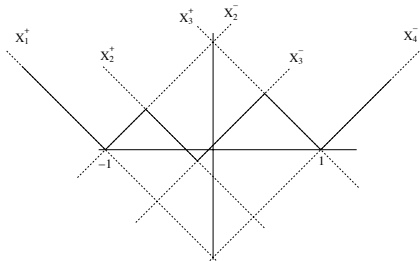
- Polygon parametrized by  $x_i$ .
- We want  $A_{min}(x_i)$

- Dependence only on cross-ratios  $\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$ .
- $N$  gluons:  $3N - 15$  of them. We want  $A_{min}\left(\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}\right)$ .
- For  $AdS_3$ , only  $N - 6$  cross-ratios. Let's start with this!

Surface on  $AdS_3$ : Ends in a 2D polygon, e.g. in the cylinder.



- Zig-zagged null polygon of  $N = 2n$  sides.
- Parametrized by  $n$  coordinates  $x_i^+$  and  $n$  coordinates  $x_i^-$ .
- We can build  $2n - 6$  invariant cross ratios.



- Minimal surfaces on  $AdS_3 \leftrightarrow$  Classical strings on  $AdS_3$ .

## Classical strings on $AdS_3$

$$\text{Strings on } AdS_3 : \vec{Y}(z, \bar{z}), \quad \vec{Y} \cdot \vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$$

$$\text{Eoms} : \partial \bar{\partial} \vec{Y} - (\partial \vec{Y} \cdot \bar{\partial} \vec{Y}) \vec{Y} = 0, \quad \text{Virasoro} : \partial \vec{Y} \cdot \partial \vec{Y} = \bar{\partial} \vec{Y} \cdot \bar{\partial} \vec{Y} = 0$$

Pohlmeyer kind of reduction  $\rightarrow$  generalized Sinh-Gordon

$$\alpha(z, \bar{z}) = \log(\partial \vec{Y} \cdot \bar{\partial} \vec{Y}), \quad p^2 = \partial^2 \vec{Y} \cdot \partial^2 \vec{Y}$$

$$\downarrow$$

$$p = p(z), \quad \partial \bar{\partial} \alpha - e^\alpha + |p(z)|^2 e^{-\alpha} = 0$$

- $\alpha(z, \bar{z})$  and  $p(z)$  invariant under conformal transformations.
- Area of the world sheet:  $\mathcal{A} = \int e^\alpha d^2z$

## Generalized Sinh-Gordon $\rightarrow$ Strings on $AdS_3$ ?

- From  $\alpha, p$  construct flat connections  $B_{L,R}$  and solve two linear auxiliary problems.

$$\begin{aligned} (\partial + B^L)\psi_a^L &= 0 \\ (\partial + B^R)\psi_{\dot{a}}^R &= 0 \end{aligned} \quad B_z^L = \begin{pmatrix} \partial\alpha & e^\alpha \\ e^{-\alpha}p(z) & -\partial\alpha \end{pmatrix}$$

### Space-time coordinates

$$Y_{a,\dot{a}} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix} = \psi_a^L M \psi_{\dot{a}}^R$$

One can check that  $Y$  constructed that way has all the correct properties.



## Two comments

### Relation to Hitchin equations

- Decompose  $B = A + \Phi$  and write flatness condition of  $B$  as

$$\begin{aligned} D_{\bar{z}}\Phi = D_z\Phi^* &= 0, & D &= \partial + [A, \cdot] \\ F_{z\bar{z}} + [\Phi, \Phi^*] &= 0 \end{aligned}$$

- Exactly the form of the so called Hitchin equations! (self-dual YM in 4d reduced to 2d)
- $A_{1,2} \rightarrow A_{1,2}$ : 2d gauge field,  $A_{3,4} \rightarrow \Phi, \Phi^*$ : Higgs field.
- We have a particular solution of the  $SU(2)$  Hitchin system.

Alternative coordinates:  $dw = \sqrt{p(z)}dz$

$$\hat{\alpha} = \alpha - \frac{1}{4} \log p\bar{p} \rightarrow \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = \sinh \hat{\alpha}, \quad \mathcal{A} = \int e^{\hat{\alpha}} d^2 w$$

- What are the restrictions on  $p(z)$  and  $\alpha(z, \bar{z})$ ?

Four cusps solution ( $n = 2$ ):  $p(z) = 1$ ,  $\hat{\alpha} = 0$

- For solutions relevant to scattering amplitudes we require  $p(z)$  to be a polynomial and  $\hat{\alpha}$  to decay at infinity.

$$p(z) = z^{n-2} + c_{n-4}z^{n-4} + \dots + c_1z + c_0$$

- Translations and re-scalings  $\rightarrow$  fix the first two coefficients.
- For a polynomial of degree  $n - 2$  we are left with  $2n - 6$  (real) variables. Exactly the number of cross-ratios for  $2n$  gluons!

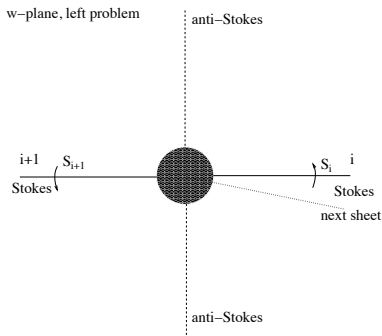
Null polygons of  $2n$  sides  $\Leftrightarrow p^{n-2}(z)$  and  $\hat{\alpha}(z, \bar{z}) \rightarrow 0$

$$\mathcal{A} = \int e^{\hat{\alpha}} d^2w \rightarrow \mathcal{A}_{reg} = \int (e^{\hat{\alpha}} - 1) d^2w$$

- Simplest case:  $p(z) = z^{n-2} \rightarrow w = \frac{n}{2} z^{n/2}$
- In the  $w$  plane we go around  $n/2$  times.

Boundary:  $|w| \gg 1 \rightarrow \hat{\alpha} \approx 0$ . Gral solution of the left problem:

$$\psi_a^L = c_a^+ \eta^+ + c_a^- \eta^-, \quad \eta^+ = \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix}, \quad \eta^- = \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$



- Focused in the left problem.
- $w$ -plane divided into two regions (anti-Stokes sectors),  $\pm \text{Re}(w) > 0$
- In each sector, one of the two solutions dominates.
- In the anti-Stokes lines, both are of the same order.

There is a Stokes phenomenon going on...

- The large solution is only defined up to a multiple of the small solution.
- Actually, as we cross the (e.g. the first) Stokes line...

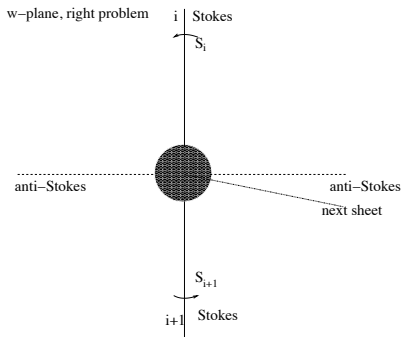
$$\begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$

- Jump in the small component of the large solution!

$$\psi_a^L = c_a^+ \eta^+ + c_a^- \eta^- \rightarrow c_a^+ \eta^+ + (c_a^- + \gamma c_a^+) \eta^-$$

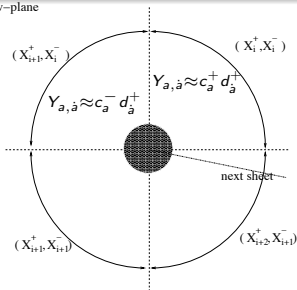
- This small component becomes important as we cross to the other anti-Stokes region.

The right-problem is similar:  $\psi_a^R = d_a^+ \left( e^{\frac{w+\bar{w}}{i}} \right) + d_a^- \left( e^{-\frac{0}{i}} \right)$

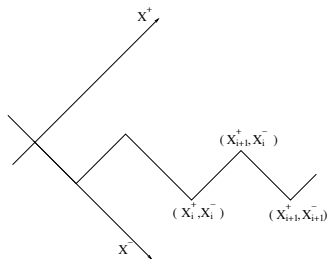


But now the Stokes and anti-Stokes lines are rotated.

w-plane



- The  $w$ -plane is divided into quadrants.
- At each quadrant, one solution of each problem ( $\eta^L$  and  $\eta^R$ ) is dominant.
- The whole region corresponds to a single point in space-time, a cusp.
- As we cross one of the anti-Stokes lines, the dominant solution L or R changes and we jump to the next cusp.
- As we go around the  $w$ -plane  $n/2$  times, we get the  $2n$  cusps!
- The general case works similarly.



## Strings on $AdS_5$ :

- Still a holomorphic quantity  $P(z) = \partial^2 \vec{Y} \cdot \partial^2 \vec{Y}$ .
- $AdS_3$  limit:  $P(z) \rightarrow p(z)^2$ .
- Two more physical fields:  $\alpha(z, \bar{z}) = \log(\partial \vec{Y} \cdot \bar{\partial} \vec{Y})$  but also  $\beta(z, \bar{z})$  and  $\gamma(z, \bar{z})$ .

For  $N$  gluons, how does the counting of cross-ratios work?

$$P(z) = z^{N-4} + c_{N-6} z^{N-6} + \dots + c_0 \rightarrow 2N - 10 \text{ real coefficients.}$$

- For  $AdS_4$  we have exactly  $2N - 10$  cross ratios, so this is the whole picture ( $\alpha$  and  $\beta$  are unique once you have fixed  $P(z)$ )
- For  $AdS_5$  there are  $N - 5$  extra degrees of freedom coming from the boundary conditions, giving the expected  $3N - 15$  cross-ratios in 4d scattering!

What about the Hitchin equations?

- Not anymore Left+Right factorization but we still have a Hitchin system!
- We obtain a particular case of  $SU(4)$  Hitchin system.

Not generic but fixed by the following projection

$$B = A + \Phi, \quad C\Phi^T C^{-1} = i\Phi, \quad CA^T C^{-1} = -A$$

Space time coordinates:

$$(\partial + A_z + \Phi_z)\psi_a = 0, \quad (\bar{\partial} + A_{\bar{z}} + \Phi_{\bar{z}})\psi_a = 0, \quad Y \approx \psi_a \Gamma_{ab} \psi_b$$



General prescription for polynomial  $p(z)$  in  $AdS_3$  or  $P(z)$  on  $AdS_5$

- Compute the space-time cross-ratios in terms of the coefficients of  $P(z)$ .
- Compute the area in terms of the coefficients of  $P(z)$ .
- Write the area in terms of the space-time cross-ratios.

First non trivial cases:

- On  $AdS_3$ :  $p(z) = z^2 - m$ , the "octagon".
- On  $AdS_5$ :  $P(z) = z^2 - U$ , the "hexagon".

but its very hard to proceed...

Better path: Be very lucky!

- Gaiotto, Moore and Neitzke have found the same Hitchin equations in a completely different context.
- They have developed a technology very useful for solving the problem at hand! (for the  $SU(2)$  case)

Idea: Use integrability to promote the Hitchin system to a family of flat connections (introduce a spectral parameter)

$$B_z^{(\zeta)} = A_z + \frac{\Phi_z}{\zeta}, \quad B_{\bar{z}}^{(\zeta)} = A_{\bar{z}} + \zeta \Phi_{\bar{z}}$$

Why is this useful?

- Consider the deformed auxiliary linear problem leading to  $Y[\zeta]$  (such that  $Y[1]$  is the physical solution).

$$(\partial + A_z + \frac{\Phi_z}{\zeta})\psi(\zeta) = 0$$

- Consider the cross-ratios as a function of  $\zeta$  (such that at  $\zeta = 1$  we obtain the physical cross-ratios).
- For  $\zeta \rightarrow 0$  or  $\zeta \rightarrow \infty$  the connections simplify and we can solve the inverse problem in a WKB approximation!

$$\psi \approx e^{\frac{1}{\zeta} \int P(z)^{1/4} dz}$$

## Setting up a Riemann-Hilbert problem.

- We expect the cross-ratios as a function of  $\zeta$  to be analytic away from  $\zeta = 0, \infty$ .
- The cross-ratios have different asymptotic behaviors as  $\zeta \rightarrow 0, \infty$  depending on the phase of  $\zeta$ , so they display Stokes phenomenon in the  $\zeta$  plane.

## Alternatively

- We can find some functions  $\chi_a[\zeta]$  of the cross-ratios, with uniform small/large  $\zeta$  behavior on the whole  $\zeta$ -plane.
- Due to Stokes, the price to pay is that there are discontinuities in the  $\chi_a[\zeta]$  along some rays in the  $\zeta$ -plane.
- GMN: This defines a Riemann-Hilbert problem which can be rewritten as an integral equation for the cross-ratios! (as a function of  $\zeta$ )

For the case of the Hexagon  $P(z) = z^2 - U^{3/4}$

$$\begin{aligned} \epsilon(\theta) = & 2|U| \cosh \theta + \frac{\sqrt{2}}{\pi} \int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')} \log(1 + e^{-\tilde{\epsilon}}) + \\ & + \frac{1}{2\pi} \int d\theta' \frac{1}{\cosh(\theta-\theta')} \log(1 + \mu e^{-\epsilon})(1 + \frac{e^{-\epsilon}}{\mu}) \\ \tilde{\epsilon}(\theta) = & 2\sqrt{2}|U| \cosh \theta + \frac{1}{\pi} \int d\theta' \frac{1}{\cosh(\theta-\theta')} \log(1 + e^{-\tilde{\epsilon}}) + \\ & + \frac{\sqrt{2}}{\pi} \int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')} \log(1 + \mu e^{-\epsilon})(1 + \frac{e^{-\epsilon}}{\mu}) \end{aligned}$$

- Exactly the form of TBA equations! ( $A_3$  or  $Z_4$  symmetric integrable theory)

- The physical cross-ratios are some simple integrals involving the solutions  $\epsilon(\theta)$ ,  $\tilde{\epsilon}(\theta)$ .
- What is the regularized area?

$$A_{reg} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta 2|U| \cosh \theta \log \left( 1 + e^{-\epsilon} \mu \right) \left( 1 + \frac{e^{-\epsilon}}{\mu} \right) + \\ + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta 2\sqrt{2}|U| \cosh \theta \log \left( 1 + e^{-\tilde{\epsilon}} \right)$$

- Exactly the free energy of the TBA system!

Some exact results...

- High temperature/Conformal limit of the TBA equations

$$U = 0 \rightarrow u_1 = u_2 = u_3$$

Hexagonal Wilson loop in  $AdS_5$  in  $U \rightarrow 0$  limit

$$R(u, u, u) = \frac{\phi^2}{3\pi} + \frac{3}{8}(\log^2 u + 2Li_2(1 - u)), \quad u = \frac{1}{4 \cos^2(\phi/3)}$$

# What have we done and what needs to be done

- We have given a further step towards the computation of classical solutions relevant to scattering amplitudes at strong coupling.
- Integrability is the key ingredient of the computation.
- More general (and neat) derivation can be given. ( L.F.A., J.

Maldacena, A. Sever, P. Vieira)

For the future...

- Could we compute these amplitudes at all values of the coupling?! see traces of this at weak coupling?
- What about other kind of solutions? e.g. correlations functions?
- Include fermions and understand non MHV amplitudes?