Yangians and Grassmannians

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Based on 1001.3348 1002.4622 [JMD, Livia Ferro]

Outline

- ✔ Superconformal + Dual superconformal $\Longrightarrow$ Yangian
- ✔ T-duality: Superconformal $\longleftrightarrow$ dual superconformal.
- ✔ Invariants of $sl(m|m)$: Grassmannian integral.
- ✔ Yangian invariance $\Longrightarrow$ unique cyclic measure $\Longrightarrow$ ACCK Grassmannian formula.

$\mathcal{N}=4$ amplitudes

On-shell $\mathcal{N}=4$ SYM is described by a PCT self-conjugate supermultiplet:

$$\Phi(\eta) = G^+ + \eta^A \Gamma_A + \frac{1}{2}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}(\eta)^4 G^-$$

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha A} = \lambda^\alpha \eta^A, \qquad \bar{q}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}.$$

Amplitudes:

$$\mathcal{A}(\Phi_1 \ldots \Phi_n) = (\eta_1)^4(\eta_2)^4 \mathcal{A}(G_1^- G_2^- G_3^+ \ldots G_4^+) + \ldots$$

$$p^{\alpha\dot{\alpha}} = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \qquad q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A, \qquad \bar{q}^{\dot{\alpha}}_A = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A}.$$

$$p\mathcal{A} = q\mathcal{A} = 0 \implies \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\mathcal{P}(\lambda, \tilde{\lambda}, \eta), \qquad \langle ij\rangle = \lambda_i^\alpha \lambda_{j\alpha}.$$

$$\mathcal{P} = \mathcal{P}^{\mathrm{MHV}} + \mathcal{P}^{\mathrm{NMHV}} + \ldots + \mathcal{P}^{\overline{\mathrm{MHV}}}.$$

Superconformal and dual superconformal symmetry

Tree-level scattering amplitudes in $\mathcal{N}=4$ SYM obey a very large symmetry algebra:

Superconformal symmetry: $j_a \mathcal{A}_n = 0$ (ignoring collinear anomalies).

$$j_a \in \left\{ p^{\alpha\dot\alpha} = \sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}, \quad k_{\alpha\dot\alpha} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}}, \quad \ldots \right\} \text{ (oscillator rep).}$$

Dual coordinates: $x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^\alpha \eta_i^A$.

Dual superconformal symmetry : $J_a \mathcal{A}_n = c_a \mathcal{A}_n$. [JMD,Henn,Korchemsky,Sokatchev]

$$J_a \in \left\{ P_{\alpha\dot\alpha} = \sum_i \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \quad K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta} x_i^{\beta\dot\alpha} \frac{\partial}{\partial x_i^{\beta\dot\beta}} + \ldots, \quad \ldots \right\}$$

e.g. dual conformal:

$$K^{\alpha\dot\alpha} \mathcal{A}_n = -\sum_i x_i^{\alpha\dot\alpha} \mathcal{A}_n$$

Yangian symmetry

Superconformal + dual superconformal $\Longrightarrow$ Yangian symmetry. [JMD,Henn,Plefka]

$$\Big(K^{\alpha\dot{\alpha}} + \sum_i x_i^{\alpha\dot{\alpha}}\Big)\mathcal{A}_n = K'^{\alpha\dot{\alpha}}\mathcal{A}_n = 0.$$

When written in terms of the on-shell superspace variables we have (c.f. [Dolan,Nappi,Witten]).

$$j_a \mathcal{A}_n = 0 \qquad j_a^{(1)}\mathcal{A}_n = 0.$$

$$j_a = \sum_i j_{ia} \qquad j_a^{(1)} = f_a{}^{cb}\sum_{i<j} j_{ib} j_{jc}.$$

These operators generate the Yangian $Y(psl(4|4))$. Zero Killing form $\Longrightarrow$ cyclicity.

In terms of twistor variables:

$$j^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

$$j^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}(-1)^{\mathcal{C}}\left[\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}}\mathcal{Z}_j^{\mathcal{C}}\frac{\partial}{\partial \mathcal{Z}_j^{\mathcal{B}}} - (j,i)\right]$$

T-duality

One can also regard the dual superconformal as the level-zero set of generators.

In this case they should be thought of as acting on $\mathcal{P}_n$ where

$$\mathcal{A}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\mathcal{P}_n, \qquad J_a \mathcal{P}_n = 0.$$

In terms of momentum twistors $\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \mu_i^{\dot\alpha} = x_i^{\alpha\dot\alpha}\lambda_{i\alpha})$: [Hodges]

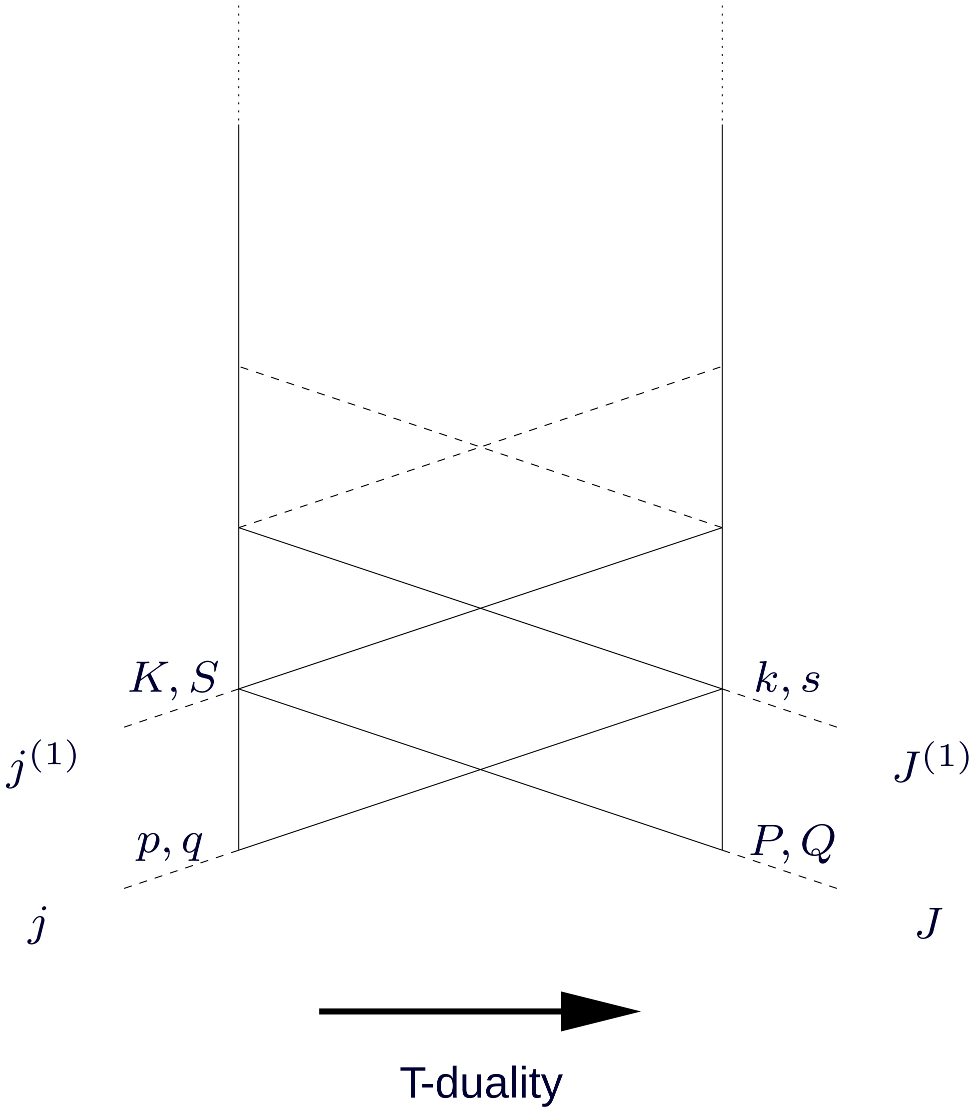
$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}$$

Original conformal symmetry $k_{\alpha\dot\alpha}\mathcal{A}_n = 0$ induces second order symmetry of $\mathcal{P}_n$,

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{\mathcal{C}} \left[\mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} - (j,i) \right].$$

so we have

$$J_a \mathcal{P}_n = 0 \qquad J_a^{(1)} \mathcal{P}_n = 0.$$



Very similar to picture of T-duality in AdS sigma model [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf],[Beisert].

Grassmannian formulas

Yangian invariants can be generated by the Grassmannian formula [Arkani-Hamed,Cachazo,Cheung,Kaplan],

$$\mathcal{L}_{n,k} = \int \frac{\prod_{a,i} dc_{ai}}{(1...k)(2...k+1)\dots(n...k-1)} \prod_a \delta^{4|4}(\sum_i c_{ai}\mathcal{Z}_i).$$

Choosing different integration contours gives different leading singularities.

There is a T-dual version [Mason,Skinner]

$$\tilde{\mathcal{L}}_{n,k} = \int \frac{\prod_{a,i} dt_{ai}}{(1...k)(2...k+1)\dots(n...k-1)} \prod_a \delta^{4|4}(\sum_i t_{ai}\mathcal{W}_i).$$

Same form up to $\mathcal{Z} \longrightarrow \mathcal{W}$ - exactly the property the general form of Yangian invariants should have.

Invariants of $Y(sl(m|m))$

What is the general form of an invariant under the $Y(sl(m|m))$ generators?

First consider the $sl(m|m)$ generators:

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} = \sum_i \left(\begin{array}{c|c} W_i^{A'} \frac{\partial}{\partial W_i^{B'}} & W_i^{A'} \frac{\partial}{\partial \chi_i^{B}} \\ \hline \chi_i^{A} \frac{\partial}{\partial W_i^{B'}} & \chi_i^{A} \frac{\partial}{\partial \chi_i^{B}} \end{array} \right)$$

Invariants come with given Grassmann degree km.

Consider Fourier tranformation $W^{A'} \longleftrightarrow \tilde{W}_{A'}$

generators $Q^A{}_{B'} = \sum_i \chi_i^A \tilde{W}_{iB'} \implies$

degree 0: need all $\tilde{W}_i = 0 \implies I_0 = \prod_i \delta^4(\tilde{W}_i) \longrightarrow 1$

degree m: need all $\tilde{W}_i$ proportional to $\tilde{W}_I \implies Q^A{}_{B'} = \tilde{W}_{IB'} Q_I^A$

$$I_1 = \delta^4(Q_I) \prod_{i \neq I} \delta^4(\tilde{W}_i - c_{Ii}\tilde{W}_I) \longrightarrow \delta^{m|m}\Big(\sum_i t_{Ii}\tilde{W}_i\Big) \longrightarrow \int dt f(t) \delta^{m|m}\Big(\sum_i t_{Ii}\mathcal{W}_i\Big)$$

Grassmannian Integral

In fact generally we have

$$I_k = \int dt f(t) \prod_{a=1}^{k} \delta^{m|m}\Big(\sum_i t_{ai}\mathcal{W}_i\Big), \qquad\qquad k \leq m.$$

Similarly one can derive this form for $k \geq n-m$.

We will assume this form for all k though we have not proven it for $m < k < n-m$.

(See talk by Emery Sokatchev).

The invariant we have found is an integral over the Grassmannian.

In general k^2 of the possible t_{ai} parameters are irrelevant - $GL(k)$ gauge symmetry.

So dimension of integral is $k \times (n-k)$:

$$\int_C K_{(k(n-k),0)}$$

What fixes the measure? - level-one generators $J^{(1)\mathcal{A}}{}_{\mathcal{B}}$.

Fixing the measure

Let us act with the level-one generators:

$$J^{(1)\,\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}\Big[\mathcal{W}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}}\mathcal{W}_j^{\mathcal{C}}\frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} - \mathcal{W}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}\Big]$$

Acting on the delta functions we can replace

$$\mathcal{W}_j^{\mathcal{C}}\frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \longrightarrow t_{ia}\frac{\partial}{\partial t_{aj}} \equiv \mathcal{O}_{ij} \qquad (gl(n) \text{ transformation})$$

So we find,

$$J^{(1)\,\mathcal{A}}{}_{\mathcal{B}} I_k = \sum_b \int dt\, f(t)\,[\mathcal{O}_b^{\mathcal{A}} - \mathcal{V}_b^{\mathcal{A}}]\,\partial_{\mathcal{B}}\delta_b \prod_{a\neq b}\delta_a \qquad \delta_a = \delta^{m|m}\Big(\sum_i t_{ai}\mathcal{W}_i\Big).$$

where

$$\mathcal{O}_b^{\mathcal{A}} = \sum_{i<j}\mathcal{W}_i^{\mathcal{A}}\mathcal{O}_{ij}t_{bi}, \qquad \mathcal{V}_b^{\mathcal{A}} = \sum_{i<j} t_{bi}\mathcal{W}_i^{\mathcal{A}}.$$

Invariance: $J^{(1)}K_{(k(n-k),0)} = d\Omega \quad \Longleftrightarrow \quad [f(t), \mathcal{O}_b^{\mathcal{A}}] = \mathcal{V}_b^{\mathcal{A}}$.

The ACCK formula

The ACCK formula is a particular solution to the invariance condition:

$$"dtf(t)" = \frac{D^{k(n-k)}t}{\mathcal{M}_1 \ldots \mathcal{M}_n}$$

$D^{k(n-k)}$ is a natural local $sl(k)$ and global $gl(n)$ invariant $k(n-k)$-form [Mason,Skinner].

$\mathcal{M}_1 = (1...k)$ etc. are the $k \times k$ minors of the matrix of the t_{ai}.

Indeed

$$[\mathcal{O}_b^{\mathcal{A}}, \mathcal{M}_p] = \left(\sum_{i=1}^{p-1} \mathcal{W}_i^{\mathcal{A}} t_{bi}\right) \mathcal{M}_p$$

so that

$$\left[\frac{1}{\mathcal{M}_1 \ldots \mathcal{M}_n}, \mathcal{O}_b^{\mathcal{A}}\right] = \mathcal{V}_b^{\mathcal{A}}.$$

so we confirm that

$$\mathcal{L}_{ACCK} = \int \frac{D^{k(n-k)}t}{\mathcal{M}_1 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{m|m}\Big(\sum_i t_{ai}\mathcal{W}_i\Big).$$

is Yangian invariant.

Uniqueness

Is the formula unique? Does there exist K' s.t. $J^{(1)}K' = d\Omega'$?

Imagine modifying the formula by some other function $f(t)$ in the integrand:

$$\mathcal{L} = \int \frac{D^{k(n-k)} t f(t)}{\mathcal{M}_1 \dots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{m|m}\Big(\sum_i t_{ai}\mathcal{W}_i\Big).$$

In order to maintain invariance we need

$$[f(t), \mathcal{O}_b^{\mathcal{A}}] = 0.$$

The operator $\mathcal{O}_b^{\mathcal{A}}$ is linear in the $\mathcal{W}_i^{\mathcal{A}}$. At most $k(n-k)$ independent vector fields:

$$\mathcal{O}_b^{\mathcal{A}} = \sum_{i=k+1}^{n} \mathcal{W}_i^{\mathcal{A}} O_{bi}.$$

The right number of constraints to fix the function $f(t)$. But - are there linear dependencies among these vector fields?

Need to evaluate

$$\det O = D^{k(n-k)} t(O_{11}, \dots, O_{k,n-k}).$$

Evaluating the determinant

To calculate $\det O$ it is simplest to fix a gauge:

$$(t_{ai}) = \left(\begin{array}{c|ccc} & t_{1k+1} & \dots & t_{1n} \\ 1_{k\times k} & \vdots & & \vdots \\ & t_{kk+1} & \dots & t_{kn} \end{array} \right).$$

The vector fields can be explicitly evaluated in these coordinates:

$$\sum_{a,j} O_{bl,aj} \frac{\partial}{\partial t_{aj}}, \qquad O_{bl,aj} = [\delta(j > l) - \delta(a \geq b)] t_{al} t_{bj}.$$

$\det O$ is a polynomial in the t_{ai} of degree $2k(n-k)$.

Easy to see that $\det O$ contains a factor of $t_{1,k+1}^2 \longrightarrow (\mathcal{M}_2)^2$.

Cyclicity of $J^{(1)} \longrightarrow$ factor of $[\mathcal{M}_1 \dots \mathcal{M}_n]^2$.

Correct degree so $\det O = c(k, n-k)[\mathcal{M}_1 \dots \mathcal{M}_n]^2$.

Simple induction shows that $c(k, n-k) \equiv 1$.

Unique cyclic measure

$\det O \neq 0$ generically $\implies$ only solution for $f(t)$ is constant.

[$\det O$ has zeros where the consecutive minors $\mathcal{M}_p = (p \dots p+k-1)$ vanish.

In principle $f(t)$ can have discontinuities across the hyperplanes where the $\mathcal{M}_p$ vanish.

In the real case one could have $|\mathcal{M}_p|$ instead of $\mathcal{M}_p$ for example.]

So the unique Yangian-invariant form on the Grassmannian is the ACCK one:

$$\mathcal{L} = \int \frac{D^{k(n-k)}t}{\mathcal{M}_1 \dots \mathcal{M}_n} \prod_a \delta^{m|m}\Big(\sum_i t_{ai} \mathcal{W}_i\Big)$$

True for all Yangians $Y(sl(m|m))$!

In the case $m = 4$ this coincides with imposing dual superconformal + ordinary conformal.

For an argument in the case of (2,2) signature see [Korchemsky,Sokatchev].

Other degrees of homogeneity

In fact we have already imposed

$$h_i I = \mathcal{W}_i^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} I = 0.$$

Relaxing this condition one finds other invariant forms for certain n, k.

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} \Big[\mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} - \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} \Big] + \tfrac{1}{2} \sum_i h_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}.$$

$$O_{bl,aj} \longrightarrow \tilde{O}_{bl,aj} = \tfrac{1}{2} [\delta(j > l) - \delta(l > j) - \delta(a > b) + \delta(b > a)] t_{al} t_{bj}$$

$$\mathrm{det} \tilde{O} = \tilde{c}(k, n-k) [\mathcal{M}_1 \ldots \mathcal{M}_n]^2$$

but now $\tilde{c}(k, n-k) = 0$ for certain $k, n-k$.

Indeed e.g. $k = 1, n = 6$:

$$\sum_{a,j} \tilde{O}_{bl,aj} \frac{\partial}{\partial t_{aj}} \Big(\frac{t_{11} t_{13} t_{15}}{t_{12} t_{14} t_{16}} \Big) = 0.$$

Summary and Outlook

Either original or dual superconformal symmetry can be thought of as fundamental.

The symmetry is the Yangian $Y(sl(4|4))$ either way.

T-duality relates the two pictures.

Chiral $sl(m|m)$ invariants given by Grassmannian integral.

There is a unique measure consistent with Yangian symmetry.

Leading singularities are given by these invariants.

Could the full planar amplitude be fixed by symmetry - including the loop integrals themselves?