Superconformal invariance of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory

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Based on work in collaboration with

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Plan of the talk

- ✔ Superamplitudes in on-shell superspace
- ✔ Dual $\mathcal{N} = 4$ superconformal symmetry
  - ✗ Holomorphic anomaly
  - ✗ NMHV superamplitudes
- ✔ Conventional conformal symmetry and twistor transform
- ✔ Grassmannian approach to (dual) superconformal invariance
- ✔ Classification of the superinvariants
  - ✗ Dual superconformal invariants in momentum twistor space
  - ✗ Properties of the Grassmannian measure
  - ✗ Twistor transform
  - ✗ Conventional conformal symmetry and uniqueness of the measure
- ✔ Conclusions and outlook

Superamplitudes in on-shell superspace I

- ✔ $\mathcal{N}=4$ gluon supermultiplet $\to$ PCT self-conjugate $\to$ holomorphic (chiral) description

  $$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \eta^A\eta^B S_{AB}(p) + \eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

  η^A ($SU(4)$ index $A = 1\dots 4$, helicity $1/2$) are Grassmann variables of on-shell superspace

- ✔ Superamplitudes $\mathcal{A}_n(\Phi(1)\dots\Phi(n))$ = expansion in powers of η_i^A

- ✔ Example: Nair's description of tree MHV amplitudes [Nair'88]

  $$\mathcal{A}_n^{\rm MHV} = \frac{\delta^{(4)}(\sum_{i=1}^n p_i)\, \delta^{(8)}(\sum_{j=1}^n \lambda_{j\,\alpha}\, \eta_j^A)}{\langle 1\,2\rangle\langle 2\,3\rangle \dots \langle n\,1\rangle}$$

- ✔ On-shell $\mathcal{N}=4$ supersymmetry
  - ✗ super-Poincaré

    $$q^A_\alpha = \lambda_\alpha \eta^A\,, \qquad \bar{q}_{A\,\dot\alpha} = \tilde\lambda_{\dot\alpha}\frac{\partial}{\partial\eta^A}\,, \qquad p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha} \;\;:\;\; \{q^A_\alpha, \bar{q}_{B\,\dot\alpha}\} = \delta^A_B\, p_{\alpha\dot\alpha}$$

  - ✗ super-conformal $\to$ non-local (2nd-order) [Witten'03]

    $$s^\alpha_A = \frac{\partial^2}{\partial\lambda_\alpha\partial\eta^A}\,, \qquad \bar{s}^A_{\dot\alpha} = \eta^A\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\,, \qquad k_{\alpha\dot\alpha} = \frac{\partial^2}{\partial\lambda^\alpha\partial\tilde\lambda^{\dot\alpha}} \;\;:\;\; \{s^\alpha_A, \bar{s}^B_{\dot\alpha}\} = \delta^B_A k_{\alpha\dot\alpha}$$

Superamplitudes in on-shell superspace II

- ✔ Invariance of the superamplitude: $p_{\alpha\dot\alpha}\mathcal{A}_n = q^A_\alpha \mathcal{A}_n = 0 \;\Rightarrow$

$$\mathcal{A}_n(\lambda,\tilde\lambda,\eta) = \delta^{(4)}(\sum_{i=1}^{n} p_i)\;\delta^{(8)}(\sum_{j=1}^{n}\;\lambda_j\,\eta_j)\Big[\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(4)} + \ldots + \mathcal{A}_n^{(4n-16)}\Big]$$

- ✔ $\mathcal{A}_n^{(4k)}(\eta)$ – homogeneous polynomials in η of degree $4k$:
  $k=0\;\to$ MHV, $k=1\;\to$ Next-to-MHV, . . . , $k=n-4\;\to\;\overline{\mathrm{MHV}}$

- ✔ Simplest case – All-order MHV superamplitude:

$$\mathcal{A}_n^{\mathrm{MHV}}(\lambda,\tilde\lambda,\eta) = \delta^{(4)}(\sum_{i=1}^{n}\;p_i)\;\delta^{(8)}(\sum_{j=1}^{n}\;\lambda_{j\,\alpha}\,\eta_j^A)\Big[\frac{M_n(p)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\Big]$$

- ✔ Define ‘ratio’ R = general/MHV superamplitude:

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}}\times\Big[R_n(\lambda,\tilde\lambda,\eta)+O(\epsilon)\Big] = \mathcal{A}_n^{\mathrm{MHV}}\;\Big[1+R_n^{(4)}+\ldots+R_n^{(4n-16)}+O(\epsilon)\Big]$$

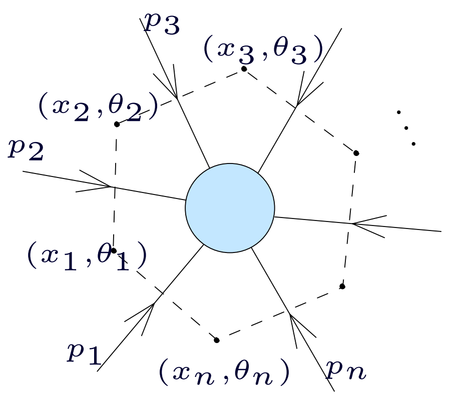
  $R_n^{(4k)}$: finite homogeneous polynomials in $\eta\to$ helicity structures and loop corrections

- ✔ Conjecture: all $R_n^{(4k)}$ are exactly dual conformal. Conformal anomaly in IR divergent MHV factor. [DHKS’08]

- ✔ Conventional superconformal at tree level, but broken by IR divergences at loop level - how?

Dual $\mathcal{N}=4$ superconformal symmetry

- ✔ Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\,\theta^A_\alpha\,,\,\lambda_\alpha)$:



  - ✗ $p=\sum_{i=1}^n p_i = 0 \;\rightarrow\quad p_i = x_i - x_{i+1}\,,\quad x_{n+1}=x_1$
  - ✗ $q=\sum_{i=1}^n \lambda_i\eta_i = 0 \;\rightarrow\quad \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i-\theta_{i+1})^A_\alpha\,,\quad \theta_{n+1}=\theta_1$

- ✔ Dual $\mathcal{N}=4$ superconformal symmetry in dual superspace [DHKS'08]
  - ✗ $\mathcal{N}=4$ super-Poincaré algebra

$$Q_{A\,\alpha}=\sum_{i=1}^n \frac{\partial}{\partial\theta_i^{A\,\alpha}}\,,\quad \bar Q^A_{\dot\alpha}=\sum_{i=1}^n \theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}\,,\; P_{\alpha\dot\alpha}=\sum_{i=1}^n\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}\;;\qquad \{Q_{A\,\alpha},\bar Q^B_{\dot\alpha}\}=\delta^B_A\,P_{\alpha\dot\alpha}$$

  - ✗ Conformal inversion: $I[x_i]=x_i^{-1}\,,\quad I[\theta_i]=\theta_i x_i^{-1}\,,\quad I[\lambda_i]=\lambda_i x_i^{-1}$

- ✔ All trees amplitudes are dual superconformal [DHKS'08] [Brandhuber et al'08] [Drummond,Henn'08]
- ✔ Fermionic T-duality of the string sigma model = dual superconformal symmetry [Alday,Maldacena'08] [Beisert,Ricci,Tseytlin,Wolf'08]

Dual superconformal symmetry: holomorphic anomaly

✔ MHV superamplitude in dual superspace

$$\mathcal{A}_n^{\rm MHV}(x,\theta,\lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1})\,\delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\,[1 + aM_n^{(1)}(x_{ij}) + O(a^2)]$$

- ✗ Tree – manifestly dual superconformal covariant.
- ✗ Loops – IR divergent factor $M_n^{(1)}(x_{ij})$ satisfies anomalous dual conformal Ward identity
- ✗ Loops – dual supersymmetry $\bar{Q}$ broken by $M_n(x_{ij})$

✔ Multi-particle discontinuity (branch cut) made from trees via unitarity:

$$\mathrm{Disc}_{x^2_{1,j+1}}\mathcal{A}_n^{\rm MHV;1} = \mathcal{A}^{\rm MHV;0}(-\ell_1,1,\ldots,j,-\ell_2)\star\mathcal{A}^{\rm MHV;0}(\ell_2,j+1,\ldots,n,\ell_1)$$

IR finite and dual conformal, but not dual supersymmetric, why?

✔ $\mathcal{A}_n^{\rm MHV;0}$ is naively holomorphic in λ, but in reality [Cachazo,Svrcek,Witten'04]

$$\frac{\partial}{\partial\tilde{\lambda}^{\dot\alpha}}\frac{1}{\langle\lambda\chi\rangle} = 2\pi\tilde{\chi}_{\dot\alpha}\delta(\langle\lambda,\chi\rangle)\delta([\tilde{\lambda},\tilde{\chi}]) \;\to\; \text{holomorphic anomaly}$$

✔ K annihilates $\langle i\, i+1\rangle^{-1}$ but $\bar{Q}$ is anomalous ! [Korchemsky,ES'09] [Bargheer et al'09]

✔ At present we have no way to control the $\bar{Q}-$anomaly:
- ✗ Duality MHV amplitudes/Wilson loops: only dual conformal, not superconformal
- ✗ If a dual supersymmetric model exists, can we imagine Poincaré SUSY breaking down ???

Dual superconformal symmetry: NMHV superamplitudes

✔ General superamplitude: $\mathcal{A}_n = \mathcal{A}_n^{\rm MHV}(a, 1/\epsilon) \left[1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\right]$

✔ Conjecture: $R_n^{(4)}$ are finite dual (super)conformal invariants [DHKS'08]

$$R_n^{(4)} = \sum_{p,q,r=1}^{n} R_{pqr} \left[1 + aM_{pqr}^{(1)}(x_{ij}) + O(a^2)\right)$$

✗ dual superconformal invariant

$$R_{pqr} = \frac{\langle q-1\, q\rangle \langle r-1\, r\rangle\, \delta^{(4)}(\langle p|x_{pq}x_{qr}|\theta_{rp}\rangle + \langle p|x_{pr}x_{rq}|\theta_{qp}\rangle)}{x_{qr}^2 \langle p|x_{pr}x_{r\,q-1}|q-1\rangle \langle p|x_{pr}x_{r\,q}|q\rangle \langle p|x_{pq}x_{q\,r-1}|r-1\rangle \langle p|x_{pq}x_{q\,r}|r\rangle}$$

✗ dual conformal invariant $M_{pqr}^{(1)}$, made of finite combinations of one-loop scalar box integrals

✗ All coefficients $= 1$, why? No known symmetry can fix them, but analytic properties do:
- Absence of spurious singularities at $\langle p|x_{pr}x_{r\,q}|q\rangle = 0$, etc. [Korchemsky,ES'09]
- Correct collinear singularities [Bargheer et al'09] [Korchemsky,ES'09]

✔ Complete N^{k}MHV $n-$particle tree found by supersymmetric BCFW recursion and shown to be dual superconformal [Brandhuber,Heslop,Travaglini'08], [Drummond,Henn'08]

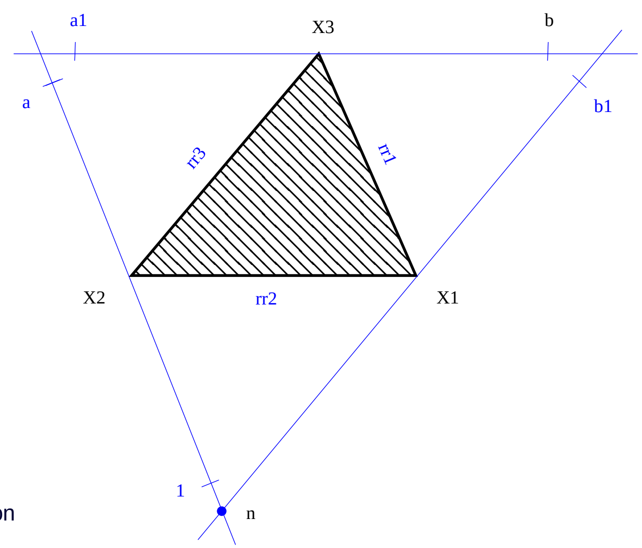
Conventional conformal symmetry and twistor transform

- ✔ Are trees both dual and conventional superconformal? Apply 2-nd-order conventional superconformal generators (not easy):
  - ✗ MHV tree is invariant [Witten'03]
  - ✗ NMHV : each R_{pqr} is invariant by itself; need analytic properties to fix coeffs [Korchemsky,ES'09]
  - ✗ N^{k}MHV : Grassmannian approach [Arkani-Hamed et al'09] [Mason, Skinner'09] [Drummond, Ferro'10]
- ✔ Alternative: do twistor ("half-Fourier") transform: $\tilde{\lambda}_{\dot{\alpha}} \to \mu^{\dot{\alpha}}$, $\eta^A \to \psi_A$, but not λ^α [Witten'03]

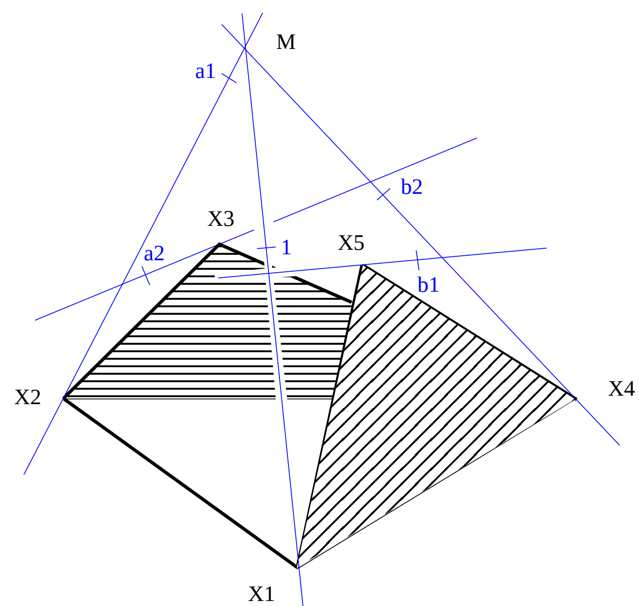
$$\mathcal{T}\left[\mathcal{A}_{n;0}^{\rm MHV}\right] = \int d^4X d^8\Theta \prod_{i=1}^{n} \frac{\delta^{(2)}(\mu_i + \langle\lambda_i|X)\,\delta^{(4)}(\psi_i + \langle\lambda_i|\Theta\rangle)}{\langle i\, i+1\rangle}$$

  - ✗ Conventional conformal generators become 1-st-order – easy to check invariance.
  - ✗ All particles lie on a line in twistor space with moduli $X_{\alpha\dot{\alpha}}$, Θ_A^α
- ✔ Twistor transform of N^{k}MHV tree: each term supported on $2k+1$ intersecting lines [Korchemsky,ES'09]

NMHV:



NNMHV:



Grassmannian approach to (dual) superconformal invariance

- ✔ Conventional superconformal symmetry made manifest as an integral of supertwistors over a Grassmannian manifold $G(n,k+2)$ [Arkani-Hamed,Cachazo,Cheung,Kaplan'09]
- ✔ Dual superconformal symmetry made manifest as an integral of momentum supertwistors over a Grassmannian manifold $G(n,k)$ [Mason,Skinner'09]
  - ✗ Momentum supertwistors [Hodges'09]

$$W = \begin{pmatrix} \lambda^{\alpha} \\ \nu_{\dot\alpha} \equiv x_{\dot\alpha\alpha}\lambda^{\alpha} \\ \chi^{A} \equiv \theta^{A}_{\alpha}\lambda^{a} \end{pmatrix} \Rightarrow \text{linear realization of dual } SU(2,2|4)$$

  - ✗ Most general dual superconformal invariant (John or “X-ray" transform)

$$R_n^k(W) = \int \frac{D^{k(n-k)}t}{\Omega(t)} \prod_{a=1}^{k} \delta^{(4|4)}\left(\sum_{i=1}^{n} t_a^i W_i\right)$$

  - ✗ Very special measure made from consecutive minors (Plücker coordinates on $G(n,k)$)

$$\Omega(t) = \prod_{i=1}^{n} T_k^{(i)}\,, \qquad T_k^{(i)} = \frac{1}{k!}\epsilon^{a_1 a_2 \dots a_k} t^{i}_{a_1} t^{i+1}_{a_2} \cdots t^{i+k-1}_{a_k}$$

  - ✗ Uniqueness theorem: only this measure assures both dual (manifest) and conventional (non-manifest) superconformal invariance [Korchemsky,ES'10] [Drummond,Ferro'10]
- ✔ Choice of contour? Leading singularity of loop integrals?

Classification of the superinvariants I

- General dual superconformal invariant in momentum twistor space
  - We are looking for chiral invariants - need Grassmann delta functions
  - Simple example: $SL(1|1)$:

$$\mathcal{Q}\, R_n(w_i,\chi_i) = \tilde{\mathcal{Q}}\, R_n(w_i,\chi_i) = 0\,, \qquad \{\mathcal{Q},\tilde{\mathcal{Q}}\} = \mathcal{C} = \text{helicity}$$

Solution:

$$\mathcal{Q}: \quad \chi_1 = 0 \Rightarrow R_n = f(w) + \chi_{\hat{\imath}} f^{\hat{\imath}}(w) + \chi_{\hat{\imath}_1}\chi_{\hat{\imath}_2} f^{\hat{\imath}_1\hat{\imath}_2}(w) + \ldots + \chi_{\hat{\imath}_1}\cdots\chi_{\hat{\imath}_{n-1}} f^{\hat{\imath}_1\cdots\hat{\imath}_{n-1}}(w)$$

$$\tilde{\mathcal{Q}}: \qquad \partial^{\hat{\imath}} f(w) = \partial^{[\hat{\imath}} f^{\hat{\imath}_1]}(w) = \ldots = \partial^{[\hat{\imath}} f^{\hat{\imath}_1\cdots\hat{\imath}_{n-2}]}(w) = 0, \qquad \text{no constraints on } f^{\hat{\imath}_1\cdots\hat{\imath}_{n-1}}(w)$$

  - Integral representation of the solution

$$R_n^k = \chi_{\hat{\imath}_1}\ldots\chi_{\hat{\imath}_k} f^{\hat{\imath}_1\cdots\hat{\imath}_k}(w) = \int Dt\; \tilde{r}_k(t) \prod_{a=1}^{k} \delta(t_a^i w_i)\delta(t_a^i \chi_i)$$

John transform for the bosonic coefficients, with redundant image $\tilde{r}_k(t)$, e.g.,

$$f^{\hat{\imath}}(w) = \int Dt\; t_1^{\hat{\imath}}\; \tilde{r}_1(t)\; \delta(t_1^1 w_1 + \sum_{\hat{\imath}=2}^{n} t_1^{\hat{\imath}} w_{\hat{\imath}})$$

  - Back to $SL(4|4)$: $R_n^k(W)$ as the top component of an invariant in a k-dimensional subspace

$$R_n^k(W) = \int [\mathcal{D}t]_{n,k} \prod_{a=1}^{k} \delta^{(4|4)}\Big(\sum_{i=1}^{n} t_a^i W_i\Big) \qquad \text{with some measure } [\mathcal{D}t]_{n,k}$$

Classification of the superinvariants II

- Properties of the Grassmannian measure $[\mathcal{D}t]_{n,k}$
  - Global $GL(n)$ and local $GL(k)$ invariance $\leftrightarrow$ Grassmannian $G(k,n)$:

$$[\mathcal{D}t]_{n,k} = \tilde{r}(t)\, D^{k(n-k)}t$$

    with the natural measure $D^{k(n-k)}t$ on $G(k,n)$ and some weight $\tilde{r}(t)$
  - The weight $\tilde{r}(t)$ is a function of locally $GL(k)$ invariant minors (global $GL(n)$ broken)

$$T_k^{(i_1,i_2,\ldots,i_k)} = \frac{1}{k!}\epsilon^{a_1 a_2 \ldots a_k} t_{a_1}^{i_1} t_{a_2}^{i_2} \cdots t_{a_k}^{i_k}$$

    made of any k columns of the $n \times k$ matrix t_a^i
  - Helicity (local rescaling of each t^i) invariance achieved if we take

$$[\mathcal{D}t]_{n,k} = \tilde{r}(t)\, D^{k(n-k)}t\,, \quad \text{with} \quad \tilde{r}(t) = \frac{\omega(t)}{T_k^{(1)} T_k^{(2)} \ldots T_k^{(n)}}$$

    with an arbitrary helicity-less and $GL(k)$ invariant function $\omega(t)$.

- What can fix the freedom in $\omega(t)$? Answer: conventional conformal symmetry imposed
  - either by second-order (non-local) conformal generators [Drummond, Ferro'10]

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}}$$

  - or by first doing a twistor transform which renders the symmetry local [Korchemsky, ES'10]

Classification of the superinvariants III

- Twistor transform

$$\mathcal{T}[\mathcal{A}_n^k] = \int [\mathcal{D}t]_{n,k} \int d^4X d^8\Theta \int \prod_{a=1}^{k} (d^2\tilde{\rho}^a d^4\xi^a)$$
$$\times \prod_{a=1}^{k} \delta^{(2)}(\sum_{1}^{n} t_a^i \lambda_i) \prod_{i=1}^{n} \frac{\delta^{(2)}(\mu_i^{\dot{\alpha}} + (X_i)^{\dot{\alpha}\alpha}\lambda_{i\,\alpha})\delta^{(4)}(\psi_{i\,A} + (\Theta_{i\,A})^{\alpha}\lambda_{i\,\alpha})}{\langle i\, i+1\rangle},$$

with “moduli space coordinates"

$$X_i = X + |\tilde{\rho}^a]\langle\rho_a^i|\,, \qquad \Theta_i = \Theta + \xi^a\langle\rho_a^i|\,, \qquad \langle\rho_a^i| \equiv \sum_{j=1}^{i-1} t_a^j\langle j|$$

- Twistor space support on intersecting twistor lines
- Conventional conformal covariance of the twistor lines equations under inversion requires that the t's are rotated by a lower-triangular $GL(n)$ matrix

$$(t_a^i)' = \sum_{j=1}^{n} t_a^j\, g_j^i(\lambda, \mu)$$

- Impose this condition, together with helicity, in infinitesimal form $\rightarrow$ Ward identities with

unique solution $\omega(t) = \text{const}$

Conclusions and outlook

- ✔ Dual superconformal symmetry is a universal feature of $\mathcal{N}=4$ scattering amplitudes
- ✔ Field-theory origin unknown (dynamical)
- ✔ AdS/CFT: fermionic T-duality is a symmetry of the string sigma model, but what is the string object dual to amplitudes with helicity? [Berkovits, Maldacena'08], [Beisert,Ricci,Tseytlin,Wolf'08]
- ✔ The closure of ordinary & dual superconformal symmetries is an infinite-dimensional Yangian $\rightarrow$ integrability? [Drummond,Henn,Plefka'09]
- ✔ Too early to say. We see that the symmetries do not completely fix even the tree. [Korchemsky,ES'09]
- ✔ The symmetries are anomalous at loop level $\rightarrow$ useless unless we can control the breaking. Not known how to do it for conventional conformal symmetry (IR breaking?)
- ✔ The MHV/Wislon loop duality does not see the helicity structure. Need to supersymmetrize the WL and test if it is dual to non-MHV superamplitudes.
- ✔ Grassmannian approach: so far looks (to me) like elegant representation theory. More to come?
- ✔ What fixes the remainder function in the MHV amplitude? Integrability? New symmetries?