The box integrals in momentum-twistor geometry

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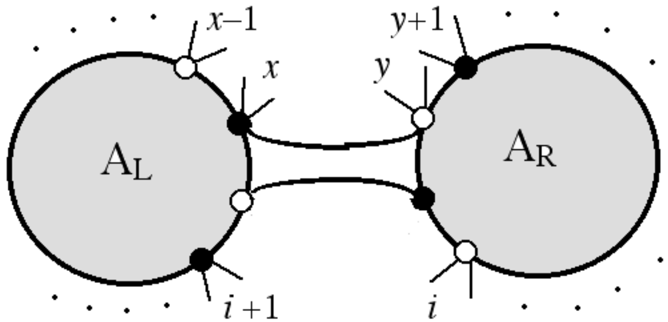
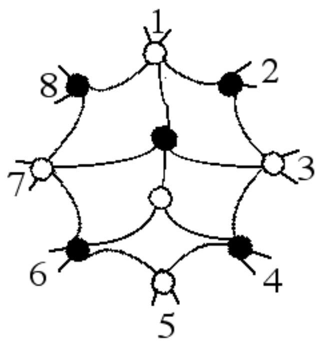
Amplitudes in supersymmetric gauge theory:

Twistor representations show conformal symmetry.

Super-BCFW joining rule becomes the super-twistor diagram joining rule
(AH, hep-th/0512336, 2005).



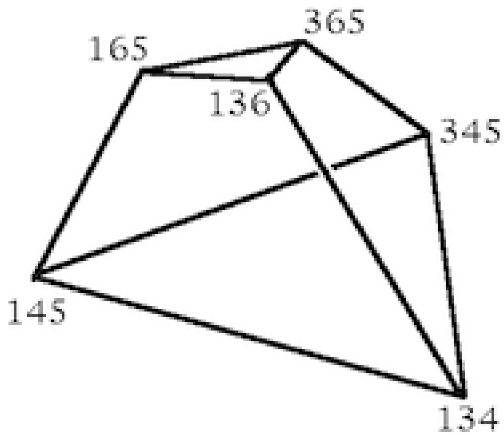
Example of a twistor diagram (part of the NNMHV 8-field amplitude):



These are disc-like structures with string-like properties.

These structures played an important part in formulating the link representation and so the Grassmannian (Arkani-Hamed, Cachazo et al.)

Manifest *dual* conformal symmetry from momentum-twistor integrals: (AH, 0905.1473, 2009).



Polytope for 6-field NMHV with spurious vertex for spurious pole.

Polytopes with dihedral symmetry for all NMHV tree amplitudes.

Also absorbed into the Grassmannian picture of N^kMHV amplitudes (Arkani-Hamed, Cachazo, Cheung, Kaplan).

Both kinds of diagrams are (super-)volumes, defined by nothing but super-boundaries.

Twistor representations also allow for the *breaking* of conformal symmetries, through the explicit appearance of the infinity twistor $I_{\alpha\beta}$.

Twistor diagrams allow for conformal-breaking boundaries at infinity, giving a detailed account of divergences and regularization.

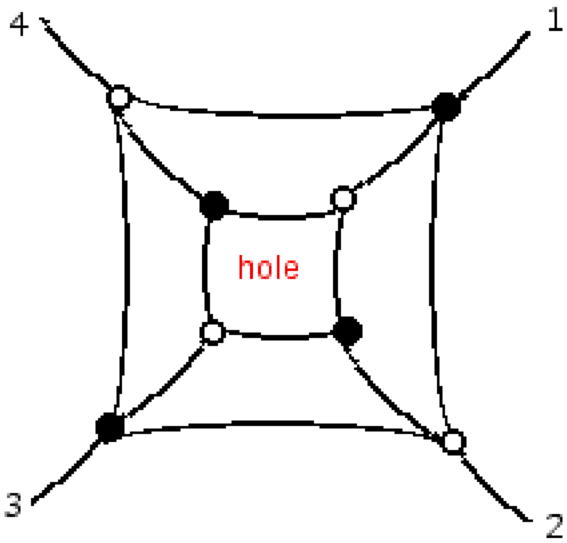
Twistor diagrams are also functionals of genuine external fields of positive/negative frequency.

Can twistor diagrams be defined for loop amplitudes?

What is the relation with the momentum-twistor integrals?

Annular twistor diagrams:

For four fields, one loop, one natural candidate:



Current investigation of higher MHV 1-loop structures, using new ideas from Arkani-Hamed, Cachazo and others.

Easier: momentum-twistor representations.

Some help from the past:

Roger Penrose (1972) set out the first twistor representation of scattering amplitudes, but infra-red and channel questions made development difficult.

The basic object of interest then was:

$$\int \frac{1}{(x-p_1)^2(x-p_2)^2(x-p_3)^2(x-p_4)^2}\, d^4x$$

for massless ϕ^4 theory. This translated naturally into a twistor integral with three contours corresponding to the three channels, giving logarithmic functions of the conformally invariant cross-ratios.

1977: AH showed a twistor integral construction leading to a *dilogarithmic* function of which all three channel amplitudes were periods.

This used linearity in CP^5 as the essential element.

2004: This dilogarithmic function is recognised as the 4-mass box integral:

$$\int \frac{1}{((p-x_1)^2+i\epsilon)((p-x_2)^2+i\epsilon)((p-x_3)^2+i\epsilon)((p-x_4)^2+i\epsilon)}\, d^4p$$

— an integral in momentum space, with x_i as region-space parameters, no two being null-separated.

How can this 1977 integral structure be recast to investigate these amplitudes?

Two main questions: (1) The shape of the contour in twistor space
(2) making contact with regularization questions.

(1) The construction requires a contour in a space of two twistors, with the form of an $S^7 \times S^1$. This corresponds precisely to the integration of the loop momentum over Wick-rotated space-time, compactified with one point at infinity to give an S^4.

The loop integration is naturally that of a twistor space, and essentially chiral.

(Important collaboration with Lionel Mason and David Skinner.)

(2) The integral we consider is actually

$$\int \frac{1}{((p-x_1)^2-\mu^2)((p-x_2)^2-\mu^2)((p-x_3)^2-\mu^2)((p-x_4)^2-\mu^2)}\,\mathrm{d}^4p$$

where μ is a regulating mass parameter. This is finite.

This integral translates into the twistor variables as:

$$\oint \frac{6\,\mathrm{d}^4 Z \wedge \mathrm{d}^4 V}{(Q_{1\alpha\beta} Z^\alpha V^\beta)(Q_{2\alpha\beta} Z^\alpha V^\beta)(Q_{3\alpha\beta} Z^\alpha V^\beta)(Q_{4\alpha\beta} Z^\alpha V^\beta)}$$

where

$$Q_i^{\alpha\beta} = 2(X_i.I)^{-1} X_i^{\alpha\beta} + \mu^2 I^{\alpha\beta}$$

is an element of the CP^5 of skew bi-twistors representing compactified CM.

The *points* of CM correspond to the the *simple* skew bi-twistors satisfying X.X=0, and the other skew bi-twistors correspond to complexified *spheres* in CM.

These are just the elements appearing in the momentum-space integral.

To perform the integral, use Feynman parameter methods, exploiting that linearity in CP^5.

The parameters span a tetrahedron (but it is useful to transform this into other shapes: cube, prism).

[Nima A.-H. and Freddy C. have taught us that pentagons are more fundamental than boxes, but these methods will extend!]

As μ tends to 0, the vertices of the tetrahedron move to the singular quadric X.X=0.

Provided no two of the x_i are null separated (the 4-mass box integral),
the limit as μ tends to 0 is finite.

Let

$$a = x_{23}^2 x_{14}^2 = K_3^2 K_1^2, \quad b = x_{12}^2 x_{34}^2 = K_2^2 K_4^2, \quad c = x_{13}^2 x_{24}^2 = st.$$

In what follows we shall write $(-\kappa)$ and $(-\tilde{\kappa})$ for the two roots of the quadratic $au^2 + (a+c-b)u + c$, so that $\kappa\tilde{\kappa} = c/a$ and $\kappa + \tilde{\kappa} = (a+c-b)/a$. We also write Δ for $a(\kappa - \tilde{\kappa})$, so that $\Delta^2 = (a+c-b)^2 - 4ac = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$.

The dilogarithmic answer is then immediate, and its symmetries geometrically natural:

$$\frac{1}{\Delta}\left\{ \begin{array}{l} \mathrm{dilog}(\kappa/\tilde{\kappa}) - \mathrm{dilog}(\tilde{\kappa}/\kappa) - \mathrm{dilog}(1-\kappa/1-\tilde{\kappa}) + \mathrm{dilog}(1-\tilde{\kappa}/1-\kappa) \\ +\mathrm{dilog}(1-\kappa^{-1}/1-\tilde{\kappa}^{-1}) - \mathrm{dilog}(1-\tilde{\kappa}^{-1}/1-\kappa^{-1}) \end{array} \right\}$$

It is *almost* conformally invariant, but broken by the Feynman prescription which applies to the individual x_i and not to their cross-ratios.

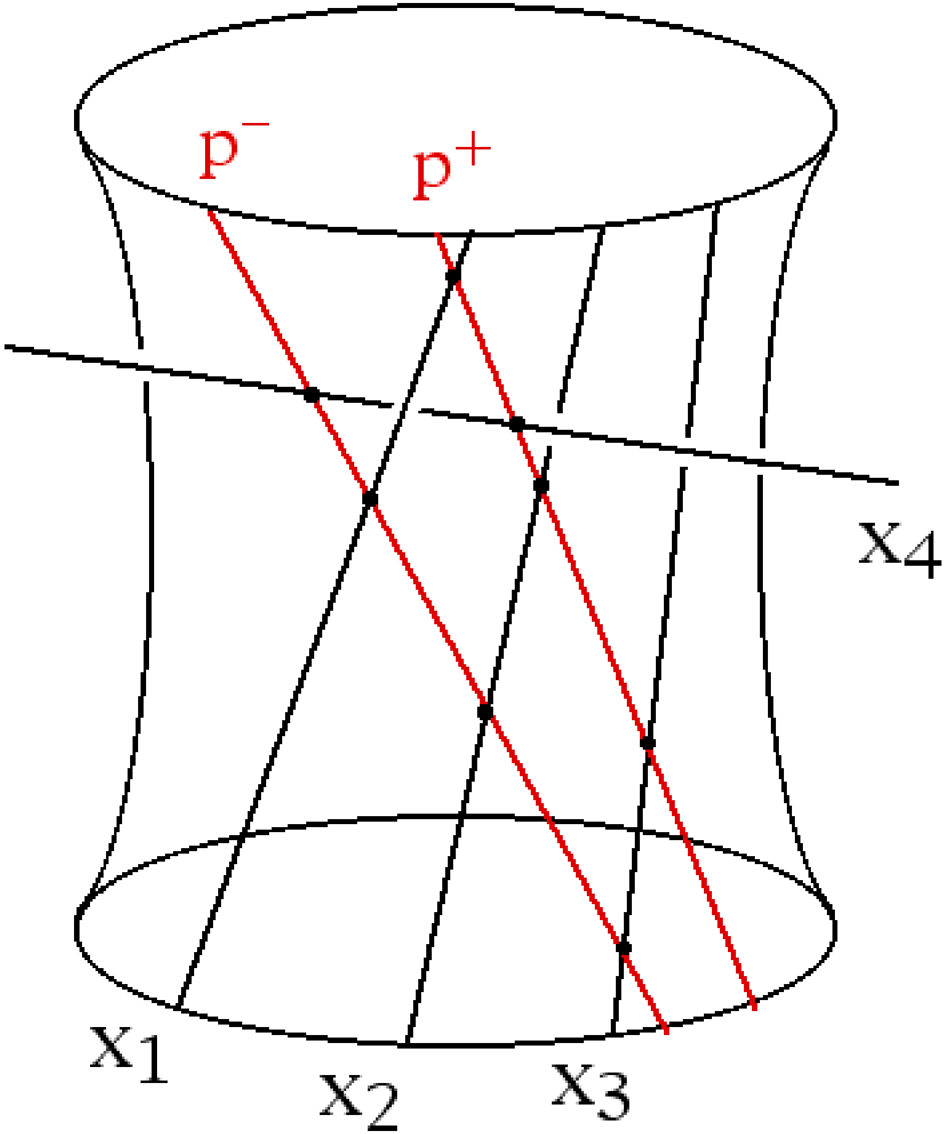
This is a remnant of the μ-regularization which is needed for the degenerate cases...

If one of the external momenta is null, the corresponding x_i are null separated and this means that a whole *edge* of the Feynman tetrahedron lies on the singular quadric. The integral is divergent as μ tends to 0.

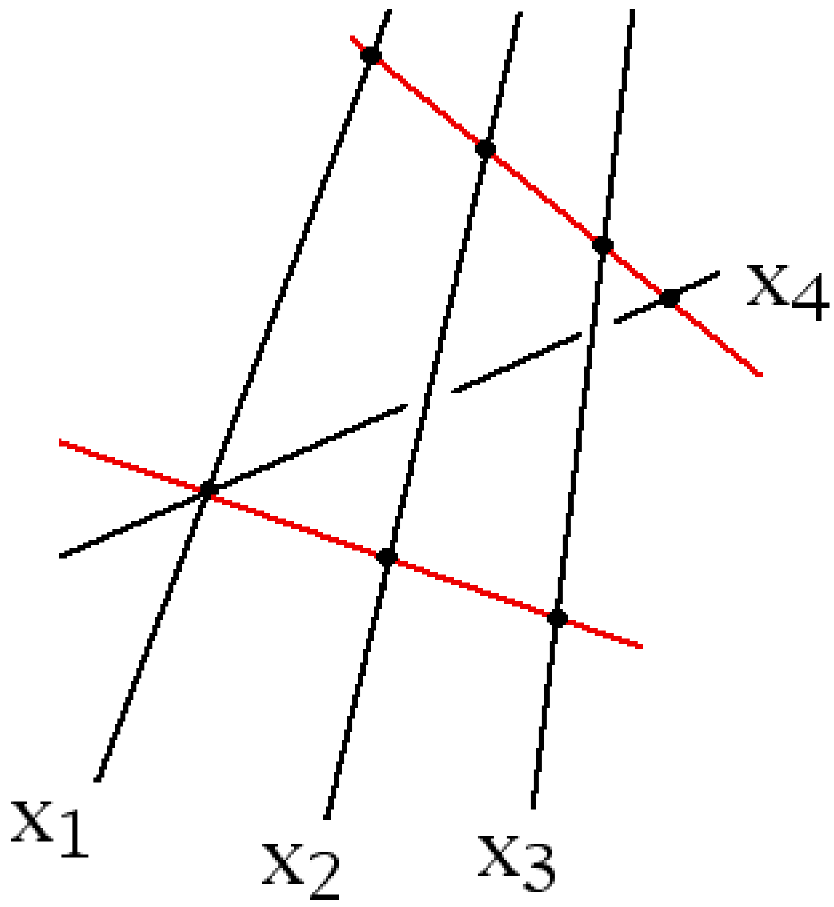
So far we have only used twistor space in the contour integration, but for considering the degenerate cases it is natural to use momentum-twistors for the external kinematic variables as well.

A null external momentum means that that the adjacent region momenta are null separated, and so they *share a momentum-twistor.*

Letting a momentum become null corresponds to letting two neighbouring momentum-twistors become equal, and merge into one.



The 4-mass integral, its geometry in twistor space. Two transversals to the four x_i.



The 3-mass integral geometry: still two transversals but now distinguished by helicity.

Note the 7 momentum twistors.

Calculation of the 3-mass box integral: elementary methods for areas and volumes.

The exact integral for finite μ is given by

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma}{(\alpha(K_2^2+\gamma s)+\beta(t+\gamma K_4^2)+\alpha\beta K_3^2-\mu^2(1+\alpha+\beta+\gamma)^2)^2}$$

Here the kinematic scalars are defined by the 7 momentum twistors and I.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma}{(\alpha(K_2^2+\gamma s)+\beta(t+\gamma K_4^2)+\alpha\beta K_3^2-\mu^2(1+\gamma)^2)^2}+O(\mu)$$

$$= \int_0^\infty \frac{\log(s+\gamma K_4^2) + \log(K_2^2+\gamma t) - \log(\mu^2 K_3^2(1+\gamma)^2)}{(t+\gamma K_4^2)(K_2^2+\gamma s) - \mu^2 K_3^2 (1+\gamma)^2} d\gamma + O(\mu)$$

$$\begin{aligned}
& \frac{1}{st - K_4^2 K_2^2} \{ \log(\mu^2 K_3^2 (st K_2^2 K_4^2)^{-\frac{1}{2}}) \log(K_2^2 K_4^2/st) \\
+ \quad & \mathrm{dilog}(1 - K_4^2 K_2^2/st) - \mathrm{dilog}(1 - st/K_4^2 K_2^2) \\
- \quad & (\mathrm{dilog}(1 - K_2^2/s) - \mathrm{dilog}(1 - t/K_4^2)) \\
- \quad & (\mathrm{dilog}(1 - K_4^2/t) - \mathrm{dilog}(1 - s/K_2^2))\}
\end{aligned}$$

The integral is finite and easy when k_1 is zero (the soft limit); now vary from this to general null k_1.

Similarly the other cases can be integrated by elementary methods corresponding to geometrical transformations of a (tetrahedral) volume.

Results agree with the standard expressions from dimensional regularization, i.e. at $\varepsilon=0$, subtracting off poles in ε.

Structure: box integral = 2 x box function x leading singularity.

In the 4-mass, 3-mass and 2-mass easy cases, the box function vanishes when the leading singularity has a pole, so the combination is finite.

The terms

$$\mathrm{dilog}(1 - K_4^2 K_2^2/st) - \mathrm{dilog}(1 - st/K_4^2 K_2^2)$$

when re-expressed in terms of momentum twistors, make their conformal invariance manifest. The other terms show actual divergence with μ, or retain only scale invariance.

Very active current work (Arkani-Hamed, Cachazo, Mason, Skinner) to locate the structure of these conformally invariant terms and interpret them as volumes.

The analysis above indicates how the regularization and conformal breaking can also be made exact.