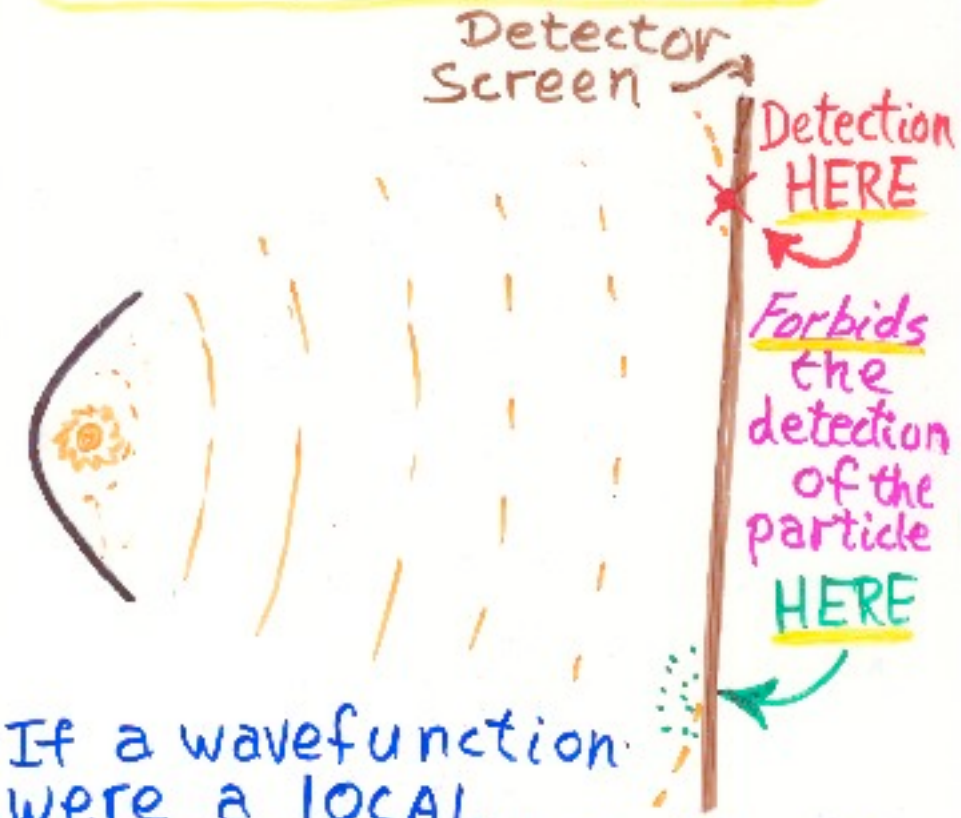
Non-Locality in the Wavefunction of a Single Particle



If a wavefunction were a LOCAL disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication

Twistor Theory

Space-time $\mathbb{M}$ (Minkowski)

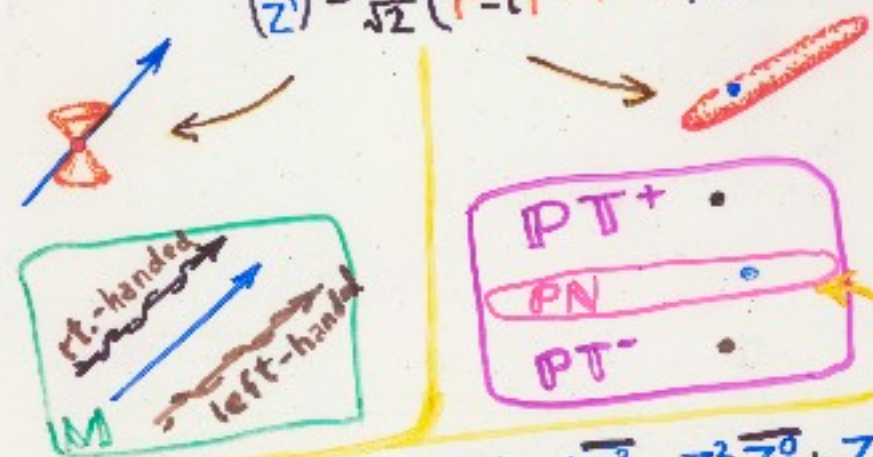
coordinates (r^0, r^1, r^2, r^3)

Twistor space $\mathbb{PN}$

homogeneous coordinates $Z^0 : Z^1 : Z^2 : Z^3$

Incidence

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$



Eqn. of $\mathbb{PN}$: $Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} = 0$

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

$$\omega^A = i \, r^{AA'} \pi_{A'}$$

$$Z^\alpha = (\omega^A, \pi_{A'})$$

incidence: $\omega^A = i r^{AA'} \pi_{A'}$

Shift of origin

$$\mathring{\omega}^A = \omega^A - i q^{AA'} \pi_{A'}$$

$$\mathring{\pi}_{A'} = \pi_{A'}$$

complex conjugate twistor

$$\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$$

Null twistor $\Rightarrow$

$$Z^\alpha \bar{Z}_\alpha = 0$$ $\leftarrow$ equation of $\mathbb{PN}$

Momentum–angular mom. for massless ptcle.

p_a = 4-momentum $\quad \left.\begin{matrix} p_a p^a = 0 \\ p_0 > 0 \end{matrix}\right\}$

M^{ab} = 6-angular momentum

Pauli-Lubanski spin vector $\quad S_a = \frac{1}{2} \epsilon_{abcd} p^b M^{cd} = s p_a$ $\quad$ $*(p \wedge M)$ $\quad$ helicity

$$p_a = \pi_A \bar{\pi}_{A'}, \quad M^{ab} = i \omega^{(A} \bar{\pi}^{B)} \epsilon^{A'B'} - i \epsilon^{AB} \bar{\omega}^{(A'} \pi^{B')}, \quad 2s = Z^\alpha \bar{Z}_\alpha$$

Note: (...) denotes symmetrization

2-spinor Notation

Vector/tensor indices: $a, b, c, \ldots, a_0, \ldots, a_1, \ldots$

4-dimensional $\quad 0, 1, 2, 3$ ← time

2-spinor indices: $\quad A, B, C, \ldots, A_0, \ldots, A_1, \ldots$ $\quad$ complex conjugate

$0, 1 \,;\, 0', 1' \quad A', B', C', \ldots, A'_0, \ldots, A'_1, \ldots$

Abstract indices:

$a = AA', \quad b = BB', \quad c = CC', \ldots, a_0 = A_0 A'_0, \ldots, a_1 = A_1 A'_1$

Standard Coordinates:

$$V^{AA'} : \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + i V^2 \\ V^1 - i V^2 & V^0 - V^3 \end{pmatrix}$$

Raise/lower indices

g_{ab}, g^{ab} ; $\quad \epsilon_{AB}, \epsilon^{AB}, \epsilon_{A'B'}, \epsilon^{A'B'}$

Symmetric $\quad$ anti-symmetric

$$g_{ab} = g_{AA'BB'} = \epsilon_{AB} \epsilon_{A'B'}$$

Interpretation of η^A:

Twistor – Strings

New development, only about 2 years old (primarily due to E. Witten) in which twistor and string ideas are combined to give new insights and calculational techniques for calculating Yang-Mills (gluon) scattering amplitudes.

At first sight, this marriage seems improbable, owing to some basic incompatibilities:

- String theory's supra-dimensionality
- Twistor theory's holomorphicity

Dimension: twistor 1+3
String 1+25, 1+9, 1+10, 2+10, 2+2

N.B.1 Spinors/twistors are complex only for signature $s-t \equiv 2 \pmod 4$ (s = space dim., t = time dim.)

$s-t \pmod 8$	0	1	2	3	4	5	6	7
number syst.	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$

N.B.2 Strings are Riemann surfaces, but p-branes not hol.

But in one version of twistor string theory one puts the strings in twistor space whose dimensions are NOT additional to those of space-time; they are "already there" according to ordinary space-time geo

Some Remarks on Twistor/Spinor conventions

- In ++++ or ++−− signature we have a complex conjugation $Z^\alpha \rightsquigarrow \bar{Z}^\alpha$ rather than $\bar{Z}_\alpha$, (Lorentzian case) so does not agree with canonical conjugation, so wavefunctions are not holomorphic.
- In ++−− we can consider δ-fns. on real α-planes. In $\mathbb{CM}$, these would be hyperfunctional objects $\therefore$ cohomology elements in space-time

Compare with Hodges elemental states

- Up/down nature of spinor parts $(\omega^A, \pi_{A'})$ fits in with conformal trans

$\hat{g}_{ab} = \Omega^2 g_{ab}$ | $\hat{\varepsilon}_{AB} = \Omega \varepsilon_{AB}$, $\hat{\varepsilon}^{AB} = \Omega^{-1}\varepsilon_{AB}$ | $\Gamma_a = \nabla_a \log \Omega$ | $\hat{\omega}^A = \omega^A$, $\hat{\pi}_{A'} = \pi_{A'} + i\Gamma_{AA'}\omega^A$

Quantum Twistor Theory

Z^α and $\bar{Z}_\alpha$ become non-commuting

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$

$$\bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar\, \delta^\alpha_\beta$$

so Z^α and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugate)

Choose $\hbar = 1$, for convenience. We find

$$p_a = \pi_A \bar{\pi}_{A'}, \qquad M^{ab} = i\omega^{(A}\bar{\pi}^{B)}\,\epsilon^{A'B'} - i\,\epsilon^{AB}\bar{\omega}^{(A'}\pi^{B')},$$

undisturbed by factor ordering, but

$$s = \frac{1}{4}\left(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha\right)$$

The standard commutators for p_a and M^{ab} follow

$$p_a p_b - p_b p_a = 0, \quad p_a M^{bc} - M^{bc} p_a = i\,(g_a{}^b p^c - g_a{}^c p^b),$$

$$M^{ab}M^{cd} - M^{cd}M^{ab} = i(g^{bc}M^{ad} - g^{bd}M^{ac} + g^{ad}M^{bc} - g^{ac}M^{bd}).$$

Lie algebra generators for the Poincaré group.

Twistor Wavefunctions

Since Z^α and $(i)\bar{Z}_\alpha$ are conjugate variables, a twistor wavefunction f ought to depend on *either* Z^α or $\bar{Z}_\alpha$, but not both. But what does it mean to say that $f(Z^\alpha)$ does not depend on $\bar{Z}_\alpha$? The condition is $\frac{\partial f}{\partial \bar{Z}_\alpha} = 0$, the Cauchy–Riemann eqns, asserting that f is holomorphic in Z^α.

Helicity eigenstates

These are eigenstates of the helicity operator $s = \frac{\hbar}{2}\left(-2 - Z^\alpha \frac{\partial}{\partial Z^\alpha}\right)$.

But $Z^\alpha \frac{\partial}{\partial Z^\alpha}$ is the Euler homogeneity operator, whose eigenstates are homogeneous functions, with eigenvalue = degree of homogeneity.

Take the integer $n = 2s/\hbar$ to represent the helicity; then f is homogeneous (holomorphic) in Z^α of $\boxed{\text{degree } -2-n}$. Alternatively use $\tilde{f}(W_\alpha)$, with $W_\alpha = \bar{Z}_\alpha$. Then $\tilde{f}$ is hol. hom. $\boxed{\text{deg. } -2+n}$

How does this relate to an ordinary space-time description?

$n=0$: $\nabla_a \nabla^a \phi = 0$

with $\nabla_a \; (= \nabla_{AA'}) = \dfrac{\partial}{\partial x^a}$

$n<0$: $\nabla^{AA'} \psi_{AB\ldots L} = 0$

$\underbrace{\psi_{AB\ldots L}}_{-n} = \psi_{(AB\ldots L)}$ symmetry brackets

$n>0$: $\nabla^{AA'} \varphi_{A'B'\ldots L'} = 0$

$\underbrace{\varphi_{A'B'\ldots L'}}_{n} = \varphi_{(A'B'\ldots L')}$

Maxwell field (complex)

$F_{ab} = \psi_{AB}\,\epsilon_{A'B'} + \epsilon_{AB}\,\varphi_{A'B'}$

Linearized gravity (complex 1st order curvature)

$K_{abcd} = \psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} + \epsilon_{AB}\,\epsilon_{CD}\,\varphi_{A'B'C'D'}$

	$S/\hbar$	hom. Z^α	hom. W_α
self-dual	+1	-4	0
anti s-d	-1	0	-4
anti s-d	-2	+2	-6
s-d	+2	-6	+2
massless neutrino	-½	-1	-3
massless anti-neutrino	+½	-3	-1
scalar wave	0	-2	-2

Free-field Maxwell equations

$$\boxed{F_{ab} = \psi_{AB}\,\epsilon_{A'B'} + \epsilon_{A'B'}\,\varphi_{A'B'}}$$

implies

$$F_{ab} = -F_{ba}$$

and

$$\nabla^{AA'}\psi_{AB} = 0, \quad \nabla^{AA'}\varphi_{A'B'} = 0$$

imply

$$\nabla_{[a}F_{bc]} = 0, \quad \nabla^a F_{ab} = 0$$

and conversely. $\boxed{dF = 0, \quad d^*F = 0}$

Linearized source-free Einstein theory

$$\boxed{K_{abcd} = \psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} + \epsilon_{AB}\,\epsilon_{CD}\,\varphi_{A'B'C'D'}}$$

implies

$$K_{abcd} = -K_{bacd} = -K_{cdab} = K_{cdab}$$

and $K_{abcd} + K_{bcad} + K_{cabd} = 0, \; K_{abc}{}^a = 0;$

moreover $\nabla^{AA'}\psi_{ABCD} = 0, \; \nabla^{AA'}\varphi_{A'B'C'D'} = 0$

imply (Bianchi) $\nabla_a K_{bcde} + \nabla_b K_{cade} + \nabla_c K_{dade} = 0$

and conversely.

Twistor description of single-photon states:

$f(Z^\alpha)$ — + helicity: hom. deg. -4

— − helicity: hom. deg. 0

$$f = \underset{s=-1}{f_0} + \underset{s=+1}{f_{-4}}$$

$$F_{ab} = \psi_{AB}\,\varepsilon_{A'B'} + \varepsilon_{AB}\,\varphi_{A'B'}$$

$$= \underset{\text{anti-self-dual}}{{}^-F_{ab}} + \underset{\text{self-dual}}{{}^+F_{ab}}$$

[abstract indices]

$$F_{ab}(x) = \oint \left\{ \varepsilon_{A'B'} \frac{\partial}{\partial\omega^{A'}}\frac{\partial}{\partial\omega^{B'}} + \varepsilon_{AB}\,\pi_{A'}\pi_{B'} \right\} f(Z^\alpha)\, d^2\pi$$

Integrate over region (dim.)

$\omega = ix\pi$ (incidence)

Integrate out ; left with $F_{ab}(x)$. Maxwell automatic

$Z^\alpha = (\omega^A, \pi_{A'})$

$d^2\pi = d\pi_{b'} \wedge d\pi_{a'}$

$= \frac{1}{2}\varepsilon^{A'B'} d\pi_{A'} \wedge d\pi_{B'}$

Contour Integral Expressions

(Whittaker, Bateman, RP, Hughston)

zero helicity:

$$\phi(x^a) = \text{con.x} \oint_{\omega = ix\pi} f(\omega^A, \pi_{A'})\, \delta\pi$$

incidence

Homogeneous version (1-dim $\oint$) $\delta\pi = \pi_{A'} d\pi^{A'}$

Inhomogeneous version (2-dim $\oint$) $\delta\pi = d\pi_{B'A} d\pi_{A'}$

Positive Helicity:

$$\varphi_{A'B'\ldots L'}(x^a) = \text{con.x} \oint_{\omega = ix\pi} \pi_{A'}\pi_{B'}\ldots\pi_{L'} f(\omega, \pi)\, \delta\pi$$

Negative Helicity:

$$\psi_{AB\ldots L}(x^a) = \text{con.x} \oint_{\omega = ix\pi} \frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\ldots\frac{\partial}{\partial\omega^L} f(\omega, \pi)\, \delta\pi$$

Canonical case (elementary state):

$$f = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$

Riemann sphere

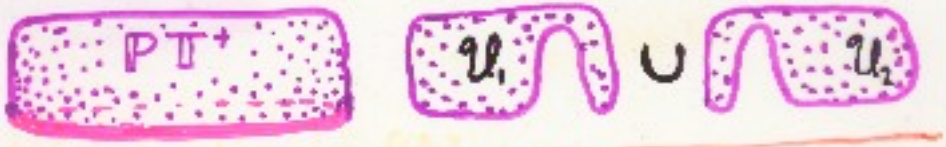
General positive frequency:

regions of singularity

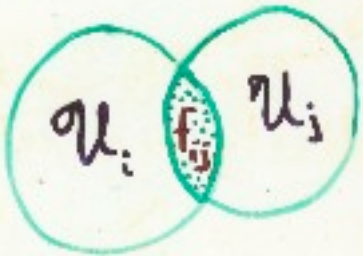
Lines in $\mathbb{P}\mathbb{T}^+$ corr. pts. in the forward tube

pts. with past-pointing imag. part

domain of def of f, whereas:



More generally, need



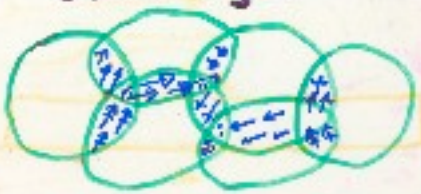
$f_{ij} = -f_{ji}$ on overlaps

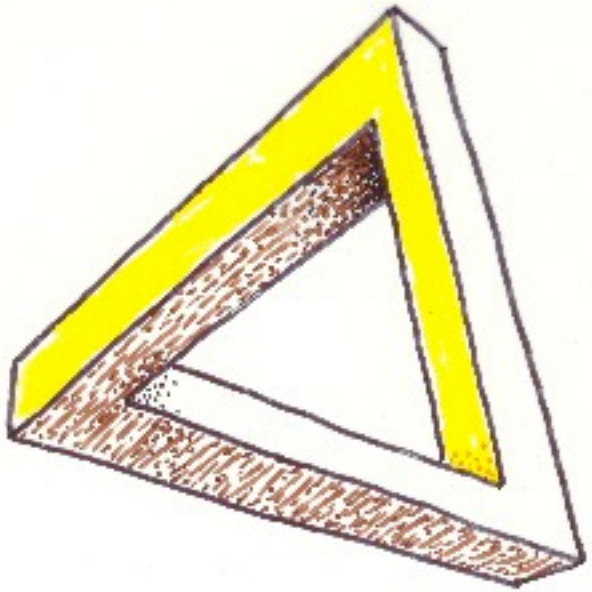
$f_{ij} + f_{jk} = f_{ik}$ on triple overlaps

$f_{ij} \equiv f_{ij} + h_i - h_j$ h_i def'd on $\mathcal{U}_i$ etc.

Branched contour $\oint$

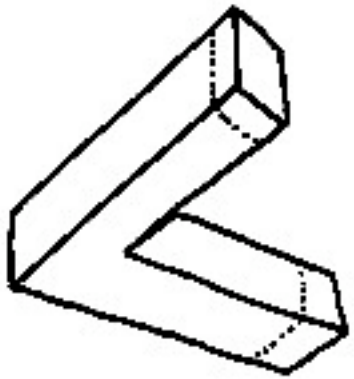
Non-linear (e.g. building a manifold)

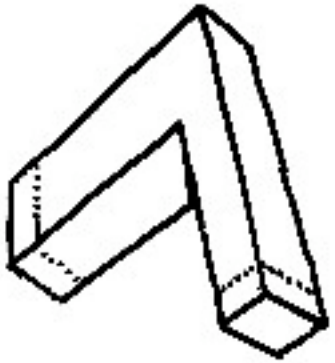


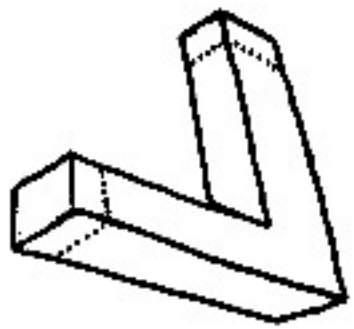


Cohomology:

a precice non-local measure — here of the degree of IMPOSSIBILITY







General relativity

Anti-self-dual (Weyl) curvature
("non-linear graviton")

$$F = I^{\alpha\beta} \frac{\partial f_{+2}}{\partial Z^\alpha} \frac{\partial}{\partial Z^\beta}$$

I "infinity twistor"

Gives general soln of asd. vacuum

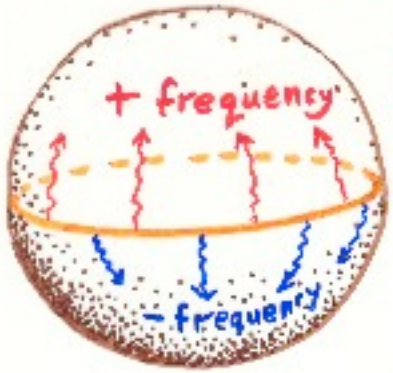
Self-dual curvature — information coded (in strange way) by "googly"

$$G = f_{-6} Z^\alpha \frac{\partial}{\partial Z^\alpha}$$

For asymptotically flat curved vacuum space-times, the full gravitational field is encoded geometrically (but strangely) in the structure of a deformed twistor space. Not yet clear how to retrieve the space-time from the twistor space.

Positive/Negative Frequency Splitting

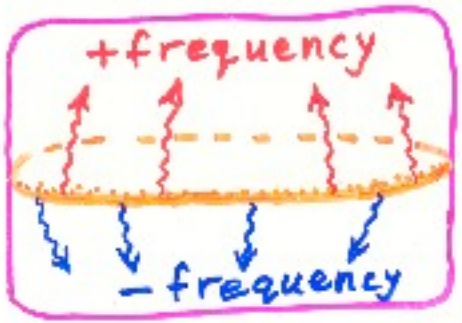
Riemann sphere $\mathbb{CP}^1$



Splitting of H^0s (ordinary functions) by holomorphic extension

real axis

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$



splitting of H^1s, by holomorphic extension

$\mathbb{PN}$ (null twistors)

This realized (finally!) one of the very early motivations behind twistor theory

Quantum Twistor Geometry

A form of non-commutative geometry?

Z^α and $\bar{Z}_\alpha$ do not commute:

$$\boxed{Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \delta^\alpha_\beta}$$

(taking $\hbar = 1$). Realized by $\bar{Z}_\alpha = -\frac{\partial}{\partial Z^\alpha}$.

Note that a satisfactory solution of the googly problem depends upon an isomorphism between the algebra (and geometry) of Z^α and that of $\frac{\partial}{\partial \bar{Z}_\alpha}$ (whatever that means!)

In effect, this seems to come down to a proper understanding of the scalar product $\langle \cdots | \cdots \rangle$, as we have:

$$\langle Z^\alpha f | g \rangle = -\langle f | \frac{\partial}{\partial Z^\alpha} g \rangle$$

and

$$\langle \frac{\partial}{\partial Z^\alpha} f | g \rangle = \langle f | Z^\alpha g \rangle .$$

Twistor functions are normally taken to be homogeneous (to represent helicity states), and for homogeneity $n > -4$ we have expressions like

$$\langle f_n | g_n \rangle = (\cdots) \oint \bar{f}_n(W_\circ) \frac{1}{(Z\cdot W)^{n+4}} g_n(Z^*) \, d^4W_\wedge d^4Z$$

(diagram 

)

But what we really need, instead of $\frac{1}{x^r}$ (writing $x = Z\cdot W$) is:

$$\frac{1}{x}, \; -\frac{1}{x^2}, \; \frac{2!}{x^3}, \; -\frac{3!}{x^4}, \; \cdots$$

to ensure $\underset{W_\circ \rightsquigarrow}{Z_\circ} \rightsquigarrow -\frac{\partial}{\partial Z^\circ}$.

But we really shouldn't insist on homogeneity (e.g. plane-polarized photon, googlys etc.). So we require $\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots$

(noting that "cross terms" of wrong homogeneity all give zero in the $\oint$)

But $\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots = F(x)$ diverges for all $x \neq 0$. Here Euler comes to our rescue!

Note (formally) $\frac{d}{dx} F(x) = F(x) - \frac{1}{x}$

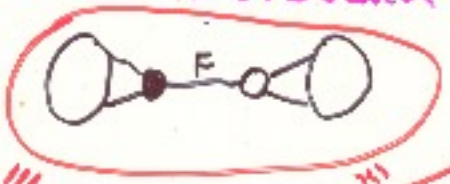
Define $G(x) = e^x \int_x^\infty \frac{e^{-y}}{y} dy$

then $\frac{d}{dx} G(x) = G(x) - \frac{1}{x}$ $\quad (u = y - x)$

also $G(x) = \int_x^\infty \frac{e^{x-y}}{y} dy = \int_0^\infty \frac{e^{-u}}{x+u} du$

Euler says: $\boxed{F(x) \equiv G(x)}$

Problem: understand equivalence



use G $\equiv$ Hodges boundary on $W\cdot Z + k$

$k = e^{-\gamma}$

Euler:

γ = Euler's constant

$$
\begin{aligned}
G(x) = & -\gamma - \log x \\
& + x(1 - \gamma - \log x) \\
& + \frac{x^2}{2!}\left(1 + \frac{1}{2} - \gamma - \log x\right) \\
& + \frac{x^3}{3!}\left(1 + \frac{1}{2} + \frac{1}{3} - \gamma - \log x\right) \\
& + \frac{x^4}{4!}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \gamma - \log x\right) \\
& + \cdots\cdots
\end{aligned}
$$

Note that 0th-order term vanishes at $\boxed{x = e^{-\gamma}}$

Hodges k

contour integral expressions can be made consistent via a "blow-down" only with boundary at $x = k$