RWQFS

After a boost in the 3^{rd} spatial direction we have $(p')^1 = p^1$, $(p')^2 = p^2$ and

$$E_{p'} = \cosh\beta\, E_p + \sinh\beta\, p^3 \qquad (1)$$

$$(p')^3 = \sinh\beta\, E_p + \cosh\beta\, p^3 \qquad (2)$$

where $E_p = \sqrt{\vec{p}^2 + m^2}$. Then, as usual,

$$\frac{d^3\vec{p'}}{2E_{p'}} = \frac{dp_1\, dp_2\, dp_3}{2E_{p'}} \left| \det \begin{pmatrix} \frac{\partial p'^1}{\partial p^1} & \frac{\partial p'^2}{\partial p^1} & \frac{\partial p'^3}{\partial p^1} \\ \frac{\partial p'^1}{\partial p^2} & \frac{\partial p'^2}{\partial p^2} & \frac{\partial p'^2}{\partial p^2} \\ \frac{\partial p'^1}{\partial p^3} & \frac{\partial p'^2}{\partial p^3} & \frac{\partial p'^3}{\partial p^3} \end{pmatrix} \right|$$

Notice that the Jacobian is upper triangular since $\frac{\partial p'^1}{\partial p^2} = 0$ etc. Then

$$= \frac{d^3\vec{p}}{2E_{p'}} \left| \frac{\partial p'^3}{\partial p^3} \right| = \frac{d^3 p}{2E_p} \left(\frac{E_p}{E_{p'}} \left| \frac{\partial p'^3}{\partial p^3} \right| \right)$$

Let's focus on the parenthesis

$$\frac{E_p}{E_{p'}}\left| \sinh\beta \frac{\partial E_p}{\partial p^3} + \cosh\beta \right| =$$

$$= \frac{E_p}{E_{p'}}\left| \sinh\beta \frac{p^3}{\sqrt{\vec{p}^2+m^2}} + \cosh\beta \right| =$$

$$= \frac{E_p}{E_{p'}} \frac{\left| \sinh\beta \, p^3 + \cosh\beta \, E_p \right|}{E_p} = 1$$

where we used (1) of the previous page!

Thus we proved that

$$\frac{d^3\vec{p}\,'}{2E_{p'}} = \frac{d^3 p}{2E_p}$$

② The field equations are

$$\frac{\partial}{\partial x^\mu}\frac{\delta S}{\delta \partial_\mu \phi} - \frac{\delta S}{\delta \phi} = 0 .$$ In our case $\Rightarrow$

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi + m^2\phi = 0$$

The conjugate variable is

$$\frac{\delta S}{\delta \partial_0 \phi} = \eta^{0\mu}\frac{\partial \phi}{\partial x^0} = \frac{\partial \phi}{\partial x^0} .$$

The canonical commutation relation are

$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = 0$$

$$\left[\phi(t,\vec{x}), \frac{\partial \phi}{\partial x^0}(t,\vec{y})\right] = i\delta^3(\vec{x}-\vec{y})$$

(3) Let's first write the mode expansion of the conjugate momentum

$$\frac{\partial \phi}{\partial x^0} = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} E_k \left[-i\, a(\vec{k})\, e^{-ikx} + i\, a^\dagger(\vec{k})\, e^{ikx} \right]$$

$$\phi = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \left[a(\vec{k})\, e^{-ikx} + a^\dagger(\vec{k})\, e^{ikx} \right]$$

Now you can insert this expressions in the commutation relation above and isolate the commutators of the modes, as done in class.

Otherwise we can invert the relations between a's and ϕ's as follows

$$\int d^3\vec{x}\, e^{i\vec{p}\vec{x}}\, \partial_0 \phi(t,\vec{x}) = \frac{E_p}{\sqrt{2E_p}} \left[-i\, a(-p)\, e^{-iE_p t} + i\, a^\dagger(p)\, e^{iE_p t} \right]$$

$$\int d^3\vec{x}\, e^{i\vec{p}\vec{x}}\, \phi(t,\vec{x}) = \frac{1}{\sqrt{2E_p}} \left[a(-p)\, e^{-iE_p t} + a^\dagger(p)\, e^{iE_p t} \right]$$

Then

$$\int d^3\vec{x}\ e^{-ipx}\frac{1}{\sqrt{2E_p}}\left[E_p\phi(t,\vec{x})-i\partial_0\phi(t,\vec{x})\right]=a^\dagger(p)$$

$$\int d^3\vec{x}\ e^{ipx}\frac{1}{\sqrt{2E_p}}\left[E_p\phi(t,\vec{x})+i\partial_0\phi(t,\vec{x})\right]=a(p)$$

Thus we can easily find the commutators for the modes

$$[a(p_1),a^\dagger(p_2)]=\int d^3\vec{x}_1\,d^3\vec{x}_2\ \frac{e^{+ip_1x_1-ip_2x_2}}{(2E_{p_1}2E_{p_2})^{1/2}}$$

$$\left(E_{p_2}\left[\phi(t,\vec{x}_1),-i\partial_0\phi(t,x_2)\right]+E_{p_1}[i\partial_0\phi,\phi]\right)$$

$$=\int d^3x_1 d^3x_2\ \frac{E_{p_1}+E_{p_2}}{(2E_{p_1}2E_{p_2})^{1/2}}\,\delta^3(\vec{x}_1-\vec{x}_2)\ e^{ip_1x_1-ip_2x_2}$$

$$=(2\pi)^3\,\delta^3(\vec{p}_1-\vec{p}_2)\,.$$

Similarly $[a(\vec{p}_1),a(\vec{p}_2)]=0$

(c) By using the mode expansion and the commutators derived before we see that

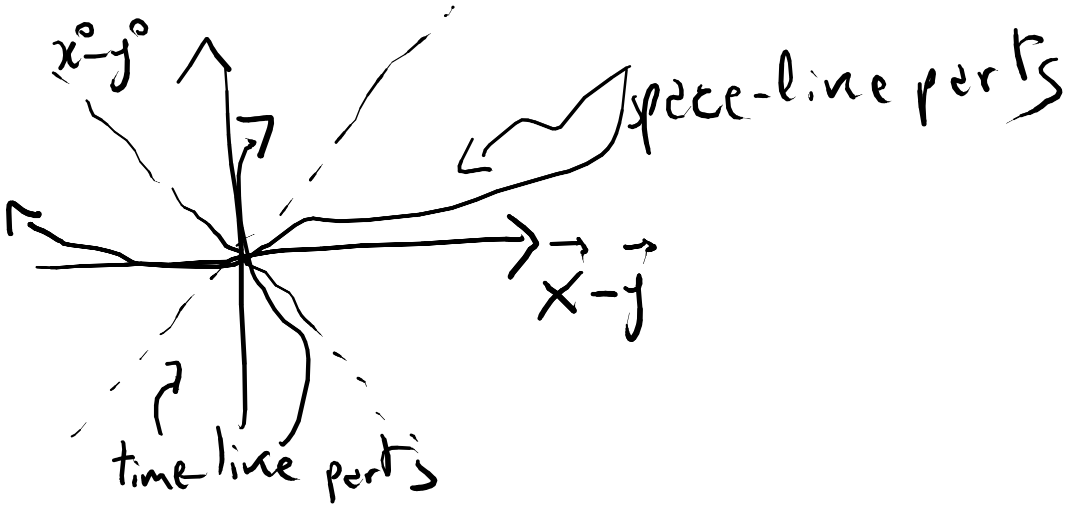
$$[\phi(x), \phi(y)] = \int \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \left(e^{-i|\vec{k}|(x^0-y^0)+i\vec{k}(\vec{x}-\vec{y})} - e^{i|\vec{k}|(x^0-y^0)-i\vec{k}(\vec{x}-\vec{y})} \right) \equiv i\Delta(x-y)$$

where I used $m^2=0$ to write $E_k = \sqrt{\vec{k}^2} = |\vec{k}|$

By changing variable $\vec{k} \to -\vec{k}$ in the second term we can write

$$\Delta(x-y) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{2|\vec{k}|} \left(e^{-i|\vec{k}|(x^0-y^0)} - e^{i|\vec{k}|(x^0-y^0)} \right)$$

From the first exercise and this way of writing $\Delta(x-y)$, we see that it is invariant under (orthocronous) Lorentz transformations. Then we can calculate $\Delta(x-y)$ in special points outside the light-cone and then derive its generic value by doing Lorentz transformations



① In the space-like case we can choose $x^0-y^0=0$ and we immediately see that $\Delta(x-y)=0$. Thus by Lorentz covariance it has to vanish in any space-like point

② In the time-like case we can choose $\vec{x}-\vec{y}=0$
Then we can write the integral over $\vec{k}$ in polar coordinates

$$d^3\vec{k} = d|\vec{k}|\,|\vec{k}|^2\,d\vartheta\,\sin\vartheta\,d\varphi$$

The integral over the angular variable is trivial when $\vec{x}-\vec{y}=0$ and gives 4π. Thus

$$\Delta(x-y) = -\int_0^{\infty} \frac{d|\vec{k}|}{(2\pi)^3 2|\vec{k}|} 4\pi |\vec{k}|^2 \left(\underbrace{e^{i|\vec{k}|(x^0-y^0)}}_{1^{st}} - \underbrace{e^{-i|\vec{k}|(x^0-y^0)}}_{2^{nd}} \right)$$

$$= -\int_{-\infty}^{+\infty} \frac{dE}{4\pi^2} E\, e^{iE(x^0-y^0)}$$

where I defined $E=|\vec{k}|$ in the 1^{st} term and $E=-|\vec{k}|$ in the 2^{nd} one. Thus

$$\Delta(x-y) = i \frac{\partial}{\partial x^0} \int_{-\infty}^{+\infty} \frac{dE}{4\pi^2} e^{iE(x^0-y^0)}$$

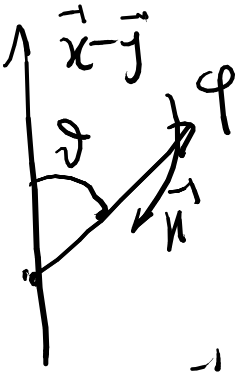
$$= \frac{i}{4\pi^2} \frac{\partial}{\partial x^0} \delta(x^0-y^0)$$

Since $\vec{x}=\vec{y}$, we have to take $x^0-y^0 \neq 0$ to stay away from the light-cone; then we see that $\Delta(x-y)=0$ also in the time-like part.

Optional

$$\langle 0|\phi(x)\,\phi(y)|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3\, 2|\vec{k}|}\, e^{-i|\vec{k}|(x^0-y^0) + i\vec{k}\cdot(\vec{x}-\vec{y})}$$

which follows in the usual way by expanding the fields in modes, using the commutation relations and $a(p)|0\rangle = 0$. By using polar coordinates for $\vec{k}$ as follows



we have

$$= \int \frac{d|\vec{k}|\,|\vec{k}|^{\cancel{2}} \sin\vartheta\, d\vartheta\, d\varphi}{(2\pi)^3\; 2\cancel{|\vec{k}|}}\, e^{-i|\vec{k}|(x^0-y^0) + i|\vec{k}|\cos\vartheta\,|\vec{x}-\vec{y}|}$$

$$= \frac{\pi}{(2\pi)^3} \int d|\vec{k}|\,|\vec{k}|\, e^{-i|\vec{k}|(x^0-y^0)} \int_{\pi}^{0} d(\cos\vartheta)\, e^{i|\vec{k}|\,|\vec{x}-\vec{y}|\cos\vartheta}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty d|\vec{n}| \; \frac{e^{-i|\vec{n}|(x^0-y^0)}}{i\,|\vec{x}-\vec{y}|} \times \frac{e^{i|\vec{n}||\vec{x}-\vec{y}|} - e^{-i|\vec{n}||\vec{x}-\vec{y}|}}{2}$$

$$= \frac{-i}{(2\pi)^2} \int_0^\infty dE \left(\frac{e^{-iE[(x^0-y^0)+|\vec{x}-\vec{y}|]}}{2|\vec{x}-\vec{y}|} - \frac{e^{-iE[(x^0-y^0)-|\vec{x}-\vec{y}|]}}{2|\vec{x}-\vec{y}|} \right)$$

$$= \frac{-i}{(2\pi)^2} \frac{1}{2|x-y|} \left(\frac{e^{-iE[+]}}{-i[+]} - \frac{e^{-i[-]E}}{-i[-]} \right)\Bigg|_{E=0}^{E=\infty}$$

where $[\pm] = [(x^0-y^0) \pm |\vec{x}-\vec{y}|]$

Now by assuming that the contribution at

$E=\infty$ vanishes (see the lectures for the proper $i\epsilon$ prescription), we get

$$= \frac{1}{(2\pi)^2}\frac{1}{2|\vec{x}-\vec{y}|}(-1)\frac{\cancel{(x^0-y^0)}-|\vec{x}-\vec{y}|-\cancel{(x^0-y^0)}-|\vec{x}-\vec{y}|}{(x^0-y^0)^2-|\vec{x}-\vec{y}|^2}$$

$$= +\frac{1}{2\pi^2}\frac{1}{(x-y)^2}$$