RWQFT 6

1) $\Sigma^{23} = \frac{i}{4}[\gamma^2, \gamma^3] = \frac{i}{2}\gamma^2\gamma^3 = \frac{1}{2}\mathbb{1}\otimes\sigma^1$

where we used Dirac's representation for the γ's

$$\gamma^j = i\sigma^2 \otimes \sigma^j \qquad j = 1, 2, 3$$

similarly we have $\Sigma^{31} = \frac{1}{2}\mathbb{1}\otimes\sigma^2$; $\Sigma^{12} = \frac{1}{2}\mathbb{1}\otimes\sigma^3$

Thus we have

$$[S^j, S^k] = \frac{1}{4}\mathbb{1}\otimes[\sigma^j, \sigma^k]$$

$$= \frac{1}{4}\mathbb{1}\otimes\sigma^l \, 2i\,\varepsilon^{jkl}$$

$$= \frac{i}{2}\varepsilon^{jkl}\left(\frac{1}{2}\mathbb{1}\otimes\sigma^l\right)$$

$$= \frac{i}{2}\epsilon^{jkl} S^l$$

2) We need to check $\partial_\mu \bar{\Psi}\gamma^\mu\Psi = 0$

$$\partial_\mu j^\mu = \partial_\mu \bar{\Psi} \gamma^\mu \psi + \bar{\Psi} \gamma^\mu \partial_\mu \psi$$

Clearly we need to use Dirac's equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 ; \quad i\partial_\mu \bar{\Psi} \gamma^\mu + m\bar{\Psi} = 0$$

Thus we have:

$$\partial_\mu j^\mu = i m \bar{\Psi} \psi - i m \bar{\Psi} \psi = 0$$

(3) In the case at hand we have:

$$i \frac{\partial \phi}{\partial t} = \frac{1}{2m} \left(-i\partial_j + qA^j\right)\left(-i\partial_j + qA^j\right) \phi +$$

$$\left(\frac{q}{2m} \sigma^j B^j - q A^0 \right) \phi$$

where $A^j = \frac{1}{2} \epsilon^{jkl} B^k x^l$. Then the 1^{st} line is

$$1^{st} \text{ line} = \frac{1}{2m} \Big[-\partial_j \partial_j \phi - iq\phi \left(\partial_j A^j\right) + q^2 A^j A^j \phi$$
$$- 2i\, q \frac{1}{2} \epsilon^{jkl} B^k x^l \partial_j \phi \Big]$$

Now $\partial_j A^j = \frac{1}{2} \varepsilon^{jkj} B^k = 0$

as ε^{jkl} is zero when two indices are equal. We'll also neglect the $A^j A^j$ term.

Then the Pauli equation reduces to

$$i \frac{\partial \phi}{\partial t} = -\frac{1}{2m} \partial_j \partial_j \phi + \frac{q}{2m} \left(\sigma^k - i \varepsilon^{klj} x^l \partial_j \right) B^k \phi$$

$$- q A^0 \phi$$

Now $S^k = \frac{1}{2} \sigma^k$ and $L^k = -i \varepsilon^{klj} x^l \partial_j$

Thus we have

$$i \frac{\partial \phi}{\partial t} = -\frac{1}{2m} \partial_j \partial_j \phi + \frac{q}{2m} \left(2 S^k + L^k \right) B^k \phi - q A^0 \phi$$

(4) $N = \int d^3 z \, \hat{\phi}^\dagger(z) \hat{\phi}(z)$

$$\hat{H} = \underbrace{\int d^3 x \, \hat{\phi}^\dagger(x) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_1(x) \right] \hat{\phi}(x)}_{H_0} +$$

$$\frac{1}{2}\underbrace{\int d^3x\, d^3y\, \hat\phi^+(x)\, \hat\phi^+(y)\, V_2(x,y)\, \hat\phi(x)\, \hat\phi(y)}_{H_1}.$$

By using the commutation relation

$$\left[\hat\phi(x)\,,\ \hat\phi^\dagger(y)\right] = \delta^3(x-y)$$

we get

$$\left[\hat N, \hat\phi(x)\right] = \int d^3z\ \hat\phi(z)\left[\hat\phi^+(z), \hat\phi(x)\right] = -\hat\phi(x)$$

$$\left[\hat N, \hat\phi^+(x)\right] = \int d^3z\left[\hat\phi(z), \hat\phi^+(x)\right]\hat\phi^+(z) = \hat\phi^+(x)$$

Notice the sign difference! Then

$$\left[\hat N, \hat\phi^{+n}(x)\,\hat\phi^n(x)\right] = n\,\hat\phi^{n}(x)\,\hat\phi^{i}(x) - n\,\hat\phi^{+n}(x)\,\hat\phi^n(x) = 0$$

for any positive integer n. Finally

$$\left[\hat N, \hat\phi^+(x)\,\nabla^2\hat\phi(x)\right] = \hat\phi^+(x)\nabla^2\hat\phi(x) + \hat\phi^+(x)\left[\hat N, \nabla^2\hat\phi(x)\right]$$

The commutator in the last term is

$$-\int d^3z\,\hat\phi(z)\,\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^j}\,\delta^3(z-x) = -\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^j}\int d^3z\,\hat\phi(z)\,\delta^3(z-x)$$

$= -\nabla^2\hat\phi(x)$. Thus $\left[\hat N, \hat H_0 + \hat H_1\right] = 0$