RWQF 5

(1) Dirac's equation reads $i\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - mc\psi = 0$

We can use Dirac's representation for the γ's

$\gamma^0 = \sigma^3 \otimes \mathbb{1}$; $\gamma^j = i\sigma^2 \otimes \sigma^j$. Then we have

$(\gamma^0)^+ = \gamma^0$, $(\gamma^j)^+ = -\gamma^j$ which can be written in a compact form as follows

$$(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0$$

(You can check this explicitely by using $(\gamma^0)^2 = \mathbb{1}_4$ and $\{\gamma^0, \gamma^j\} = 0$). Then by taking the hermitean conjugate of Dirac's equation we have

$$(i\gamma^\mu \partial_\mu \psi - mc\psi)^+ = 0 \Rightarrow -i\partial_\mu \psi^+ (\gamma^\mu)^+ - mc\psi^+ = 0$$

$\Rightarrow i\partial_\mu \psi^+ \gamma^0 \gamma^\mu \gamma^0 - mc\psi^+ = 0$. Finally multiply this identity by γ^0 from the right

$$i\partial_\mu \psi^+ \gamma^0 \gamma^\mu (\gamma^0)^2 - mc\psi^+ \gamma^0 = i\partial_\mu \bar{\psi} \gamma^\mu + mc\bar{\psi} = 0$$

(2) $\bar{\psi}'\gamma^5\psi'(x') = (\psi')^\dagger\gamma^0\gamma^5\psi'(x') =$

$= \psi^\dagger P^\dagger \gamma^0\gamma^5 P \psi(x)$

Thus let us focus on the matrix product

$P^\dagger\gamma^0\gamma^5 P = (\gamma^0)^\dagger \gamma^0\gamma^5\gamma^0 = (\gamma^0)^2\gamma^5\gamma^0$

where I used $(\gamma^0)^\dagger = \gamma^0$ as seen in the previous exercise. We also have $(\gamma^0)^2 = \mathbb{1}_4$, so

$P^\dagger\gamma^0\gamma^5 P = \gamma^5\gamma^0 = -\gamma^0\gamma^5$ where I used

$\gamma^0\gamma^5 = \gamma^0 i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^1\gamma^2\gamma^3$

$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^2\gamma^3$

which implies $\gamma^0\gamma^5 = -\gamma^5\gamma^0$ or $\{\gamma^0,\gamma^5\} = 0$.

Thus by using this result for $P^\dagger\gamma^0\gamma^5 P$ we have

$\bar{\psi}'\gamma^5\psi'(x') = \psi^\dagger(-\gamma^0\gamma^5)\psi(x) = -\bar{\psi}\gamma^5\psi(x)$

(b) Let us divide the (orthocronous) Lorentz transformations Λ in those that can be written as $\Lambda = e^{\varepsilon^\mu{}_\nu}$ where $\eta_{\rho\mu}\varepsilon^\mu{}_\nu = -({}^t\varepsilon)_{\nu\rho}$ and all the others. For transformations of the first type we have

$$\Lambda_s = e^{-\frac{i}{2}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}} \quad \text{with} \quad \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Now the key point is that the property $\{\gamma^5, \gamma^\mu\} = 0$ implies $[\gamma^5, \gamma^\mu\gamma^\nu] = 0$

$$\Uparrow$$

$$\gamma^5\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^5\gamma^\nu = +\gamma^\mu\gamma^\nu\gamma^5$$

This implies $[\gamma^5, \Sigma^{\mu\nu}] = 0$ and $[\gamma^5, \Lambda_s] = 0$ (Just expand the exp in the definition of Λ_s).

Then for this transformations we have

$$\Lambda_s^{-1}\gamma^5\Lambda_s = \Lambda_s^{-1}\Lambda_s\gamma^5 = \gamma^5$$

We also have $\det \Lambda = 1$ (the determinant of Λ is always either $+1$ or -1 and to check the sign you can work in the $\varepsilon \to 0$ limit

$$\det e^{\varepsilon^\mu{}_\nu} \sim \det e^0 = \det \mathbb{1} = 1).$$

The other type of transformation are generated by combining a parity transformation with the boost / rotations we saw:

$$\Lambda = P e^{\varepsilon^\mu{}_\nu} \quad \text{on vectors where } P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Lambda_s = \gamma^0 e^{-\frac{i}{2} \varepsilon_{\mu\nu} \Sigma^{\mu\nu}}$$ where as saw earlier

in the exercise γ^0 is the spinor representation of P. Then we have $\det \Lambda = \det P \cdot \det e^{\varepsilon^\mu{}_\nu} = -1$

and by using $\{\gamma^0, \gamma^5\} = 0$ $\quad \{\Lambda_s, \gamma^5\} = 0$

Then for this type of transformation we have

$$\Lambda_s^{-1} \gamma^5 \Lambda_s = - \gamma^5 = \gamma^5 \det \Lambda$$

[comment: in principle we should check also non-orthocronous transformations, but we did not cover time-reversal in class].

③ $\Sigma^{12} = \frac{i}{4}[\gamma^1, \gamma^2] = \frac{i}{2}\gamma^1\gamma^2 = -\Sigma^{21}$

In the Dirac representation

$$\gamma^1\gamma^2 = (i\sigma^2 \otimes \sigma^1)(i\sigma^2 \otimes \sigma^2) = -i\, \mathbb{1} \otimes \sigma^3$$

Then the corresponding form for Λ_S is

$$\Lambda_S = e^{-\frac{i}{2}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}}$$ where in our case

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\pi & 0 \\ 0 & -2\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since it acts only in the x^1, x^2 plane and $\vartheta = 2\pi$. Thus we have

$$\Lambda_S = e^{-\frac{i}{2} 2\pi (\Sigma^{12} - \Sigma^{21})} = e^{-i2\pi \frac{i}{2}\gamma^1\gamma^2} =$$

$$e^{\pi\gamma^1\gamma^2} = e^{-i\pi(\mathbb{1} \times \sigma^3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$