RWQF3

(1) The Klein-Gordon equation is

$$0 = \hbar^2 \frac{\partial^2}{\partial t^2}\phi(t,x^i) - \hbar^2 c^2 \frac{\partial^2}{\partial x^{i2}}\phi(t,x^i) + m^2c^4\phi(t,x^i)$$

Now by substituting $\phi(t,x^i) = N e^{-\frac{imc^2}{\hbar}t}\psi(t,x^i)$

we get

$$\hbar^2 \frac{\partial}{\partial t}\left[-\frac{imc^2}{\hbar} e^{-\frac{imc^2}{\hbar}t}\psi(t,x^i) + e^{-\frac{imc^2}{\hbar}t}\frac{\partial}{\partial t}\psi(t,x^i)\right] - \hbar^2c^2$$

$$e^{-\frac{imc^2}{\hbar}t}\frac{\partial}{\partial x^{i2}}\psi(t,x^i) + m^2c^4 e^{-i\frac{mc^2}{\hbar}t}\psi(t,x^i) = 0$$

One more step:

$$-mc^2 \cancel{e^{-\frac{imc^2}{\hbar}t}}\psi(t,x^i) - 2i\hbar mc^2 \cancel{e^{-\frac{imc^2}{\hbar}t}}\frac{\partial}{\partial t}\psi(t,x^i) +$$

$$+\hbar^2 \cancel{e^{-\frac{imc^2}{\hbar}t}}\frac{\partial^2}{\partial t^2}\psi(t,x^i) - \hbar^2c^2 \cancel{e^{-\frac{imc^2}{\hbar}t}}\frac{\partial^2}{\partial x^{i2}}\psi(t,x^i) +$$

$$+m^2c^4 \cancel{e^{-\frac{imc^2}{\hbar}t}}\psi(t,x^i) = 0$$

$$\hbar \frac{\partial}{\partial t}\psi(t,x^i) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^{i2}}\psi(t,x^i) + \frac{\hbar^2}{mc^2}\frac{\partial^2}{\partial t^2}\psi(t,x^{i})$$

If $\psi(t,x^i)$ is slowly varying in time, i.e.

if $\hbar \frac{\partial}{\partial t}\psi(t,x) \ll mc^2\psi(t,x)$, then

we get the usual Schrödinger equation.

Let's now check that $\phi_m(t,x) = N e^{-\frac{i}{\hbar}mc^2 t}$

is a solution of the K.G. equation.

$$\hbar \frac{\partial^2 \phi_m}{\partial t^2} = \hbar^2 N\left(-\frac{i}{\hbar}mc^2\right)^2 e^{-\frac{i}{\hbar}mc^2 t} = -(mc^2)^2\phi_m$$

$$\hbar^2 \frac{\partial^2 \phi_m}{\partial (x^i)^2} = 0 \quad \text{for any } i = 1,2,3$$

The by plugging these results in the first eq. of the previous page, we see that the K.G. eq. is indeed satisfied.

Let's now perform a boost, for instance, along the x^1 direction. This is described by the $SO(1,3)$ matrix Λ

$$\Lambda = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$$

where $\Lambda^{-1} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then $\phi'(x') = \phi(x)$ implies

$$\phi'(x') = e^{-\frac{i}{\hbar} mc x^0} = e^{-\frac{i}{\hbar} mc (\Lambda^{-1})^0{}_\nu (x')^\nu}$$

$$= e^{-\frac{i}{\hbar} mc \left(\cosh\beta\, (x')^0 - \sinh\beta\, (x')^1\right)}$$

By inserting this $\phi'(x')$ in the K.G. eq we have

$$\frac{\partial^2 \phi'}{\partial (x'^0)^2} = -(mc^2)^2 \cosh^2\beta \, \phi'$$

$$\frac{\partial^2 \phi'}{\partial (\phi'^1)^2} = -(mc^2) \sinh^2\beta \, \phi'$$

Then the K.G. eq is a consequence of

$$\cosh^2\beta - \sinh^2\beta = 1 \quad \forall \beta .$$

(2) We need to check what components of $\vec{L}$ commute with the Hamiltonian.

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \; ; \; \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_z .$$

Then by using the standard commutation relations

$$[\hat{x}, \hat{p}_x] = i\hbar \;, \quad [\hat{x}, \hat{y}] = [\hat{x}, \hat{z}] = [\hat{y}, \hat{z}] = 0$$

$$[\hat{x}, \hat{p}_y] = [\hat{x}, \hat{p}_z] = 0$$

$$[\hat{y}, \hat{p}_y] = i\hbar \;, \quad [\hat{y}, \hat{p}_x] = [\hat{y}, \hat{p}_z] = 0$$

$$[\hat{z}, \hat{p}_z] = i\hbar \;, \quad [\hat{z}, \hat{p}_x] = [\hat{z}, \hat{p}_y] = 0$$

$$[\hat{p}_x, \hat{p}_y] = [\hat{p}_x, \hat{p}_z] = [\hat{p}_y, \hat{p}_z] = 0$$

we get $[\hat{L}_z, \hat{H}] = 0$, while the other two possibilities $[\hat{L}_y, \hat{H}] \neq 0$ and $[\hat{L}_z, \hat{H}] \neq 0$

$$[\hat{L}_z, \hat{H}] = \left[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x \,,\; \underbrace{\frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right)}_{A} + \underbrace{\frac{m\omega^2}{2}\left(\hat{x}^2 + \hat{y}^2\right)}_{B}\right]$$

is zero because the two parts A and B vanishes seperately.

(A) $\left[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right)\right]$

(The term with $\hat{p}_z$ does not contribute since $\hat{p}_z$ commute with $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$). Then

$$\left[\hat{x}\hat{p}_y, \hat{p}_x^2 + \hat{p}_y^2\right] = \hat{p}_x \hat{p}_y 2i\hbar$$

$$\left[-\hat{y}\hat{p}_x, \hat{p}_x^2 + \hat{p}_y^2\right] = -\hat{p}_y \hat{p}_x 2i\hbar$$

This implies (A) vanishes. Similarly

$$\left[\hat{x}\hat{p}_y, \hat{x}^2 + \hat{y}^2\right] = \hat{x}\hat{y} 2(-i\hbar)$$

$$\left[-\hat{y}\hat{p}_x, \hat{x}^2 + \hat{y}^2\right] = -\hat{y}\hat{x} 2(-i\hbar)$$

implies that (B) vanishes

The potential term $\frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$ is responsible for $\left[\hat{L}_x, \hat{H}\right] \neq 0$ and $\left[\hat{L}_y, \hat{H}\right] \neq 0$

$$\left[\hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{x}^2 + \hat{y}^2\right] = -\hat{z}\hat{p}_y 2(-i\hbar)$$

implies $\left[L_x, \hat{H}\right] = 2i\hbar \frac{1}{2} m\omega^2 \hat{z}\hat{p}_y$

and similarly for $\left[\hat{L}_y, \hat{H}\right] = -2i\hbar \frac{1}{2} m\omega^2 \hat{z}\hat{p}_x$

A rotation along the x axis is described by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then we have for ψ'

$$\psi' = e^{-iEt\, ip\left(\sin\theta\, y' + \cos\theta\, z'\right)}$$

$$e^{-\frac{m\omega}{2\hbar}\left[x^2 + \left(\cos\theta\, y' + \sin\theta\, z'\right)^2\right]}$$

We can recognize that the two parts

of ψ satisfy separately

$$\frac{\hat{p}_z^2}{2m}\psi = \frac{-1}{2m}\frac{\partial^2}{\partial z^2}\psi = \frac{p^2}{2m}\psi$$

$$\left[\frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2\right)\right]\psi = \hbar\omega\,\psi$$

Thus from the standard Schrödinger eq we get

$$\frac{p^2}{2m} + \hbar\omega = E$$

On the contrary $\psi'(x)$ is clearly not a solution of the original equation. [If you wish to check this explicitly the best approach is to expand ψ' in the small ϑ-limit and focus on the term linear in ϑ; that term will not satisfy the S.E. and is easier to comput than the full answer!]