RWQF2

① First let's invert the relations defining $\hat{a}, \hat{a}^+$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^+) \quad ; \quad \hat{p} = \sqrt{\frac{m\hbar\omega}{2}}\;\frac{\hat{a}-\hat{a}^+}{i}$$

Then we have

$$\hat{H} = \frac{1}{2\cancel{m}}\,\frac{\cancel{m}\hbar\omega}{2}\left(-\hat{a}^2 - \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) +$$

$$\frac{1}{2}\cancel{m}\,\omega^{\cancel{2}}\,\frac{\hbar}{2\cancel{m}\cancel{\omega}}\left(\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right)$$

$$= \frac{1}{2}\hbar\omega\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Also $i\hbar = [\hat{x},\hat{p}] = \frac{\hbar}{2i}\left[\hat{a}+\hat{a}^+,\ \hat{a}-\hat{a}^+\right] =$

$$= \frac{\hbar i}{2}\left([a,\hat{a}^+] - [\hat{a}^+,\hat{a}]\right) = i\hbar\,[\hat{a},\hat{a}^+]$$

which implies $[\hat{a},\hat{a}^+] = 1$

In the Heisenberg picture we have

$$\hat{a}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\,\hat{a}_H(0)\,e^{-\frac{i}{\hbar}\hat{H}t}$$

where $\hat{a}_H(0) \equiv \hat{a}$ and $\hat{H}$ is $\hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})$.

$$\frac{d}{dt}\left(\hat{a}_H(t)\right) = \frac{d}{dt}\left(e^{\frac{i}{\hbar}\hat{H}t}\,\hat{a}_H(0)\,e^{-\frac{i}{\hbar}\hat{H}t}\right)$$

$$= \frac{i}{\hbar}\hat{H}\left(\dots\right) - \left(\dots\right)\frac{i}{\hbar}\hat{H}$$

where the parenthesis stay for $e^{\frac{i}{\hbar}\hat{H}t}\hat{a}_H(0)e^{-\frac{i}{\hbar}\hat{H}t} = \hat{a}_H(t)$

Then $\displaystyle\frac{d}{dt}\left(\hat{a}_H(t)\right) = +\frac{i}{\hbar}\left[\hat{H}, \hat{a}_H(t)\right]$

By taking the hermitean conjugate we have

$$i\hbar\frac{d\hat{a}_H(t)}{dt} = \left[\hat{a}_H(t), \hat{H}\right] \Rightarrow i\hbar\frac{d\hat{a}_H^{+}(t)}{dt} = \left[\hat{a}_H^{+}(t), \hat{H}\right]$$

② We can always represent a boost with

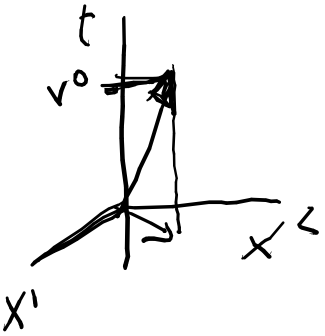
$$\begin{pmatrix} (v')^0 \\ (v')^1 \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \end{pmatrix}$$

The $(v')^0 = \cosh\beta\, v^0 + \sinh\beta\, v^1$

If v is timelike then $v^2 \equiv v^\mu \eta_{\mu\nu} v^\nu < 0$

This implies $|v^0| > \sqrt{v^i v^i}$ and since $v^0 > 0$

$v^0 > \sqrt{v^i v^i} = |\vec{v}|$. This just means that the projection of v along t is bigger than its projection in the space-like plane



Then $(v')^0 \geq \cosh\beta\, v^0 - |\sinh\beta\, v^1| \geq$

$\cosh\beta\,(v^0 - v^1) \geq \cosh\beta\,(v^0 - |\vec{v}|)$

which is positive by hypothesis. A time reversal transformation $x^0 \to -x^0$, $x^i \to x^i$ satisfies ${}^t\Lambda\,\eta\,\Lambda = \eta$, but maps v_0 in $-v_0$.

(3) $$\psi'_\mu(x') = \frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial}{\partial x'^\mu}\phi(x)$$

where I used the property $\phi'(x') = \phi(x)$. Then

$$\psi'_\mu(x') = \frac{\partial}{\partial x'^\mu}\phi(x(x')) = \frac{\partial \phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu}$$

By comparing these eqs. we see that

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

Then if $(x')^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x^\nu = (\Lambda^{-1})^\nu{}_\mu (x')^\mu$

$$\Rightarrow \frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu .$$ Then

$$\psi'_\mu(x') = (\Lambda^{-1})^\nu{}_\mu \frac{\partial \phi}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \psi_\nu$$

Notice that $(\Lambda^{-1})^\nu{}_\mu = \eta^{\nu\nu'}({}^t\Lambda)_{\nu'}{}^{\mu'}\eta_{\mu'\mu}$ since

$$(\eta\,{}^t\Lambda\,\eta)\Lambda = \eta({}^t\Lambda\eta\Lambda) = \eta\cdot\eta = \mathbb{1}$$

in matrix notation

$$\epsilon^{\mu\nu\rho} A'_\nu(x')\, \psi'_\rho(x') =$$

$$\left[\epsilon^{\mu\nu\rho} (\Lambda^{-1})^{\nu'}{}_{\nu} (\Lambda^{-1})^{\rho'}{}_{\rho}\right] A_{\nu'}(x)\, \psi_{\rho'}(x) =$$

$$\frac{1}{\det \Lambda} \Lambda^{\mu}{}_{\mu'} \epsilon^{\mu'\nu'\rho'} A_{\nu'}(x)\, \psi_{\rho'}(x)$$

The final step can be checked explicitly for $\mu=0$, $\mu=1$ and $\mu=2$. An easier way is to write

$$\epsilon^{\mu\nu\rho} = \epsilon^{\sigma\nu\rho} (\Lambda)^{\mu}{}_{\mu'} (\Lambda^{-1})^{\mu'}{}_{\sigma}$$

since $(\Lambda\Lambda^{-1})$ is just the identity matrix. Then

$$[\dots] = \underbrace{\epsilon^{\sigma\nu\rho} (\Lambda^{-1})^{\mu'}{}_{\sigma} (\Lambda^{-1})^{\nu'}{}_{\nu} (\Lambda^{-1})^{\rho'}{}_{\rho}}_{*} \Lambda^{\mu}{}_{\mu'} A_{\nu'} \psi_{\rho'}$$

This combination vanishes if any two indices in the set μ', ν', ρ' are equal. If $\mu=0$, $\nu=1$, $\rho=2$, then $*$ is just the definition of $\det(\Lambda^{-1})$. For the other choices only the overall sign can change and we have $* = \det(\Lambda^{-1})\, \epsilon^{\mu'\nu'\rho'}$