RNQF 1

(1)

a) Let's consider the variation $\delta\psi^*$

$$\frac{\partial}{\partial t}\frac{\delta \mathcal{L}}{\delta \frac{\partial \psi^*}{\partial t}} + \frac{\partial}{\partial x^i}\frac{\delta \mathcal{L}}{\delta \frac{\partial \psi^*}{\partial x^i}} - \frac{\delta \mathcal{L}}{\delta \psi^*} = 0$$

$$0 \quad -\frac{1}{2\mu}\frac{\partial}{\partial x^i}\frac{\partial \psi}{\partial x^i} - i\frac{\partial \psi}{\partial t} + V\psi = 0$$

which yields $\quad i\frac{\partial \psi}{\partial t} = -\frac{1}{2\mu}\frac{\partial^2 \psi}{\partial x^{i2}} + V\psi = 0$

The other variation $(\delta\psi)$ leads to the the complex conjugate version of the e.o.m.

$$i\frac{\partial}{\partial t}\psi^* - \frac{1}{2\mu}\frac{\partial}{\partial x^i}\frac{\partial \psi^*}{\partial x^i} + V\psi^* = 0$$

b) If ϑ is constant then

$$\psi^*\frac{\partial}{\partial t}\psi \to \psi^* e^{-i\vartheta}\frac{\partial}{\partial t}e^{i\vartheta}\psi = \psi^*\frac{\partial}{\partial t}\psi$$

and similarly for the space derivatives. Thus S is unchanged.

c) If we make ϑ t-dependent we have

$$S[\psi e^{i\vartheta}] = \int dt\, d^3\vec{x} \left\{ \psi^* e^{-i\vartheta(t)} \frac{\partial}{\partial t}\left(e^{i\vartheta(t)}\psi\right) - \frac{1}{2\mu}\frac{\partial\psi^*}{\partial x^i}\frac{\partial\psi}{\partial x^i} - V|\psi|^2 \right\}$$

$$= S[\psi] + \int dt \left[\int d^3\vec{x}\, |\psi|^2\right]\left(i\frac{\partial}{\partial t}\vartheta(t)\right) + \ldots$$

The object in the square parenthesis is the conserved charge

d) We should check $\frac{d}{dt}\int d^3\vec{x}\, |\psi|^2 = 0$ when the e.o.m. are satisfied.

$$\frac{d}{dt}\int d^3x\, |\psi|^2 = \int d^3x \left[\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right]$$

$$= i \int d^3x \left[\left(-\frac{1}{2\mu} \frac{\partial^2 \psi^*}{\partial x^{i2}} + V\psi^* \right) \psi - \psi^* \left(-\frac{1}{2\mu} \frac{\partial^2 \psi}{\partial x^{i2}} + V\psi \right) \right]$$

where we used the e.o.m. Then

$$= i \int d^3x \, \frac{\partial}{\partial x^i} \left(-\frac{1}{2\mu} \frac{\partial \psi^*}{\partial x^i} \psi + \psi^* \frac{\partial \psi}{\partial x^i} \right)$$

(The object in the parenthesis is the current j).

Each term in this result is the integral of a total derivative. For instance for $i=1$ we have

$$= i \int d^3x \, \frac{\partial}{\partial x^1} (\dots) = i \int dx^2 dx^3 \, (\dots) \Big|_{x^1=-\infty}^{x^1=\infty}$$

which vanishes since $\psi(x_1 \to \pm\infty, x^2, x^3) \to 0$ (otherwise $\int |\psi|^2 d^3x$ is divergent!).

2

a) $(\sigma^1)^\dagger = \sigma^1$; i.e. σ^1 is an hermitean matrix, thus it can represent a Q.M. observable.

b) $$e^{i\vartheta\sigma^1} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\vartheta\sigma^1)^n$$

Let me first notice that $(\sigma^1)^2 = 1$;

So $(\sigma^1)^{2n} = \mathbb{1}$ and $(\sigma^1)^{2n+1} = \sigma^1$. Then

$$e^{i\vartheta\sigma^1} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n \vartheta^{2n}\,\mathbb{1} +$$

$$\sum_{n=0}^{\infty} \frac{i}{(2n+1)!}(-1)^n \vartheta^{2n+1}\,\sigma_1$$

$$= \begin{pmatrix} \cos\vartheta & i\sin\vartheta \\ i\sin\vartheta & \cos\vartheta \end{pmatrix} = U$$

c) U is unitary ; i.e. $U^\dagger U = U U^\dagger = \mathbb{1}$

3

a) $$[A, BC] = ABC - BCA =$$

$$= [A,B]\,C + B\cancel{AC} - \left(B[C,A] - \cancel{BAC}\right)$$

$$= [A,B]\,C + B\,[A,C]$$

b) $$[[A,B],C] = [AB, C] - [BA, C]$$

By using the result obtained in a), we have

$$= A[B,C] + [A,C]B - B[A,C] - [B,C]A$$

$$= [A,[B,C]] + [[A,C],B]$$

Alternatively

$$= ABC - ACB + ACB - CAB - BAC + BCA - BCA + CBA$$

$$= A\{B,C\} - \{A,C\}B + \{B,C\}A - B\{A,C\}$$

$$= \{A,\{B,C\}\} - \{\{A,C\},B\}$$