MSci EXAMINATION

PHY-415 (MSci 4242)   Relativistic Waves and Quantum Fields

Time Allowed: 2 hours 30 minutes

Date: 13/5/2009

Time: 14:30

Instructions:

Answer THREE QUESTIONS only. Each Question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

CANDIDATES SHOULD NOTE THAT THE EXAMINATION AND ASSESSMENT REGULATIONS STATE THAT POSSESSION OF UNAUTHORISED MATERIALS AT ANY TIME WHEN A CANDIDATE IS UNDER EXAMINATION CONDITIONS IS AN ASSESSMENT OFFENCE. PLEASE CHECK YOUR POCKETS NOW FOR ANY NOTES THAT YOU MAY HAVE FORGOTTEN THAT ARE IN YOUR POSSESSION. IF YOU HAVE ANY THEN PLEASE RAISE YOUR HAND AND GIVE THEM TO AN INVIGILATOR NOW.

EXAM PAPERS CANNOT BE REMOVED FROM THE EXAM ROOM.

Data:

We use units where $\hbar = c = 1$. A **FORMULA SHEET** is provided at the end of the examination paper.

Examiners:

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YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Question 1: The Dirac equation (set $\hbar = c = 1$):

(a) Derive the continuity equations of the Klein-Gordon equation and of the Dirac equation. Hence, derive the associated densities ρ and currents $\vec{j}$ for these two wave equations. Explain how the continuity equation implies conservation of the quantitiy $\int d^3x\rho$. For which of the two wave equations can ρ be interpreted as a probability density? Explain which property of the wave equations is responsible for this. What is the correct field theoretic interpretation of ρ for the other wave equation?

[7]

(b) Find all plane wave solutions of the Dirac equation using the Dirac matrices given on the formula sheet for a Dirac particle at rest $\vec{p} = 0$. State the physical properties and interpretations of the different solutions. State two alternative methods to generate solutions with arbitrary spatial momentum $\vec{p}$.

[5]

(c) Consider the covariant form of the Dirac equation. Assume that Ψ transforms under a Lorentz transformation as $\Psi(x) \rightarrow \Psi'(x') = S(\Lambda)\Psi(x)$, with $x' = \Lambda x$ and $S(\Lambda)$ a four-by-four matrix. Show that the Dirac equation is form invariant (i.e. covariant) if

$$S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda) = \Lambda^{\nu}{}_{\mu}\gamma^{\mu}.$$

[5]

(d) Consider adding a term proportional to $F_{\mu\nu}\gamma^{\mu}\gamma^{\nu}\Psi$ to the covariant form of the Dirac equation, where $F_{\mu\nu}$ is the electromagnetic field strength. Is such a term compatible with the covariance and gauge invariance of the Dirac equation? Which physical property of the electron would such a term change?

[3]

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Question 2: Dirac equation in an electromagnetic field (set $\hbar = c = 1$):

(a) In classical relativistic mechanics the interaction of a particle carrying charge q in an external electromagnetic field is obtained by substituting the 4-momentum as $p^\mu \rightarrow p^\mu + qA^\mu$, where A^μ denotes the electromagnetic 4-vector potential. Hence, find the covariant and the Hamiltonian form of the Dirac equation for a Dirac fermion with charge q in an external electromagnetic field.

[**4**]

(b) Show that the electromagnetic field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under the gauge transformation $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$, with Λ an arbitrary, real function of the space-time coordinates. How must the Dirac wavefunction Ψ transform under a gauge transformation, in order that the combined transformation of A^μ and Ψ preserves the form of the Dirac equation in the presence of an electromagnetic field derived in (a), up to an overall phase factor? Use this result to show that the QED Lagrangian $\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\Psi}(i\slashed{\partial} + q\slashed{A} - m)\Psi$ is gauge invariant.

[**7**]

(c) The non-relativistic limit of the Dirac equation in the presence of an external electromagnetic field is given by the Pauli equation

$$i\frac{\partial \phi}{\partial t} = \left(\frac{(\widehat{\vec{p}} + q\vec{A})^2}{2m} + \frac{q}{2m}\vec{\sigma} \cdot \vec{B} - qA^0 \right)\phi \,,$$

where ϕ is a two-component spinor wavefunction. Consider a weak and homogeneous magnetic field with vector potential $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$ where $\vec{B}$ is constant. Neglecting terms quadratic in $\vec{B}$ show that the Pauli equation can be written in the form

$$i\frac{\partial \phi}{\partial t} = \left(\frac{(\widehat{\vec{p}})^2}{2m} + \frac{q}{2m}(\widehat{\vec{L}} + 2\widehat{\vec{S}}) \cdot \vec{B} - qA^0 \right)\phi \,.$$

[**9**]

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Question 3: Massless Dirac particles:

In the following use the *chiral representation* of the Dirac matrices

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \, , \quad \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad i = 1, 2, 3 \, ,$$

where the σ^i denote the Pauli matrices.

(a) Consider positive and negative energy, plane wave solutions of the Dirac equation (using the above Dirac matrices) in the massless case $(m = 0)$

$$\Psi = e^{\mp i p \cdot x} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \, ,$$

where ϕ and χ denote two component column spinors. Derive equations for ϕ and χ.

[6]

(b) Define the helicity of a particle. What is the form of the helicity operator for a Dirac particle? In part (a) of this question you derived equations for ϕ and χ. Find the helicities of ϕ and χ for positive and negative energy.

[4]

(c) The transformation of a Dirac spinor under a Lorentz transformation is given by $\Psi \rightarrow S(\Lambda)\Psi$. For a rotation around the z-axis the transformation is given by the matrix

$$S = e^{-i\theta\sigma(J_z)} \, , \quad \text{where } \sigma(J_z) = \frac{i}{4} \left[\gamma^1, \gamma^2 \right] \, .$$

Find the matrix S.   [5]

Consider now two particular cases of the positive energy solutions considered in part (a) of this question. In both cases set $\chi = 0$ and solve the equation for ϕ derived in part (a) for the particular 4-momenta $p^\mu = (E, E, 0, 0)$ and $p^\mu = (E, 0, E, 0)$. This corresponds to massless Dirac particles moving in the x- and y-direction respectively. (Hint: set one of the components of ϕ equal to one and solve for the other). Show that, up to an overall constant, the four component spinors $\begin{pmatrix} \phi \\ \chi \end{pmatrix}$ for the two cases are related by the matrix S derived above if you choose $\theta = \pi/2$.

[5]

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Question 4: The Klein-Gordon field

(a) The **free, complex** Klein-Gordon field ϕ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi) \ .$$

Obtain the field equation for ϕ and $\phi^\dagger$ and the Hamiltonian density $\mathcal{H}$ in terms of ϕ, $\phi^\dagger$ and their derivatives.

[4]

State without proof Noether's theorem for internal symmetries. Show that the transformation $\phi \to e^{i\alpha}\phi$, with α constant, is a symmetry of the Lagrangian and derive the associated conserved current j_μ. Use the field equations to verify that $\partial_\mu j^\mu = 0$.

[4]

In the following questions you may *assume* the commutation relations

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = (2\pi)^3 2E_{\vec{k}} \delta^{(3)}(\vec{k} - \vec{k}') \ , \quad \text{and} \quad \left[a(\vec{k}), a(\vec{k}')\right] = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right] = 0 \ .$$

(b) The **free, neutral** Klein Gordon field $\phi = \phi^\dagger$ may be expanded in the form

$$\phi = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x}\right] \ ,$$

with $E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$ and $k \cdot x = E_{\vec{k}} t - \vec{k} \cdot \vec{x}$. Calculate the commutator of two field operators at general space-time points x and y

$$i\Delta(x - y) = [\phi(x), \phi(y)] \ .$$

Note that you do not have to perform the final three-momentum integral explicitly. Show that the result is Lorentz invariant and vanishes for space like separations $(x - y)^2 < 0$. Discuss the implication of the latter property of $[\phi(x), \phi(y)]$ for causality.

[6]

(c) Calculate the expectation value of the product of two field operators $iD(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle$. You do not have to perform the final three-momentum integral explicitly. Does $iD(x - y)$ vanish for $(x - y)^2 < 0$? Show that $iD(x - y)$ is a solution of the Klein-Gordon equation.

[6]

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Question 5: The S-matrix

(a) Assume that the Hamiltonian operator H is split up into a free and interacting part as $H = H_0 + H_{int}$. The interaction Hamiltonian in the interaction picture is given as

$$H_I \equiv e^{iH_0t}(H_{int})_S e^{-iH_0t}$$

Show that a state $|\psi(t)\rangle_I$ in the interaction picture obeys the Schrödinger equation

$$i\frac{d|\psi(t)\rangle_I}{dt} = H_I(t)|\psi(t)\rangle_I .$$

(Note that states and operators in the Schrödinger picture (subscript S) and in the interaction picture (subscript I) are related as: $|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$ and $\mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}$.)

[**5**]

(b) Write the solution of the Schrödinger equation in Question 5(a) as

$$|\psi(t)\rangle_I = U(t, t_0)\,|\psi(t_0)\rangle_I ,$$

where $U(t, t_0)$ is the unitary time evolution operator with $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$. Hence, find a differential equation for $U(t, t_0)$ and from that the integral representation of this equation imposing the condition $U(t, t) = 1$. The solution to this equation is given by Dyson's formula

$$U(t, t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)$$

where T is the time ordering operator. Show that Dyson's formula is a solution to the differential equation you derived for $U(t, t_0)$ with the boundary condition $U(t, t) = 1$.

[**10**]

(c) State how the S-matrix (operator) S is related to the operator $U(t, t_0)$. Give a definition of initial and final states in a scattering process and write S-matrix elements or scattering amplitudes in terms of S, initial states and final states. Give a qualitative description of how physical cross sections are obtained from a given S-matrix element or scattering amplitude.

[**5**]

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Formula Sheet (in units $\hbar = c = 1$)

<u>4-vector notation:</u>

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} = a_\mu b_\nu g^{\mu\nu} \ \text{ with } \ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \ , \ \ x_\mu = (t, -\vec{x})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \ , \ \ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right) \ , \ \ \widehat{p}^\mu = i\partial^\mu \ , \ \ \widehat{p}_\mu = i\partial_\mu$$

<u>Klein-Gordon equation:</u> $(-\widehat{p} \cdot \widehat{p} + m^2)\psi = (\partial_\mu \partial^\mu + m^2)\psi = (\Box + m^2)\psi = 0$

<u>Free Dirac equation</u> in <u>Hamiltonian form:</u> $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)\Psi$, or in <u>covariant form:</u>

$$(i\slashed{\partial} - m)\Psi = (i\gamma^\mu \partial_\mu - m)\Psi = (\widehat{\slashed{p}} - m)\Psi = (\gamma \cdot \widehat{p} - m)\Psi = (\gamma^\mu \widehat{p}_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$(\alpha^i)^2 = \mathbb{I}, \, i = 1, 2, 3; \ \beta^2 = \mathbb{I}; \ \alpha^i \alpha^j + \alpha^j \alpha^i = 0 \, , \, i \neq j; \ \alpha^i \beta + \beta \alpha^i = 0 \, , \, i \neq j;$$
$$\gamma^0 = \beta, \ \gamma^i = \beta \alpha^i, \ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{I} \, ,$$
$$\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

Dirac representation:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \, i = 1, 2, 3 \ , \ \ \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} ,$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Note that α^i, β and γ^0 are Hermitian, wheras the γ^i are anti-Hermitian.

End of Examination Paper